Doctoral Dissertation

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Abstract

We develop the theory of strong reductions in the reverse mathematics zoo and show by means of novel tree labeling constructions that while $\mathsf{D}^2_2 \leq_{\mathrm{sW}} \mathsf{SIPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SRT}^2_2$, $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SIPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{D}^2_2$.

Effective Techniques in Reverse Mathematics

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Effective Techniques in Reverse Mathematics

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1 Introduction and Background

1.1 Reverse Mathematics and Its Zoo

This dissertation develops and extends the theory of strong computable reductions in reverse mathematics.

Reverse mathematics is a metamathematical program which maps out the logical landscape of mathematics by asking of a given theorem which axioms are strictly necessary to prove that theorem. Traditionally, this cartography has been carried out by comparing a given theorem against each of several standard benchmark collections of axioms—subsystems of second order arithmetic, to be precise—to find the weakest benchmark subsystem capable of proving the given theorem and the strongest benchmark subsystem which can be proved from the given theorem, with all proofs carried out over the base theory RCA_0, a theory which—modulo some technical concerns—corresponds to constructive mathematics. Quite often these two benchmarks are the same, that is, the given theorem is logically equivalent to one of the standard benchmark subsystems over the weak base theory. This leads to the "Big Five phenomenon" that very many theorems of ordinary mathematics are equivalent to one of five subsystems of second-order arithmetic. Until fairly recently this phenomenon constituted the big picture of reverse mathematics (see Simpson [22] for a standard account), but now a growing body of research in reverse mathematics concerns itself with the relationships between those theorems that do not fall into the Big Five pattern.

A naturally occurring example of such a theorem is Ramsey's theorem for pairs, that theorem of combinatorics which states that if we color every pair of natural numbers either red or blue (i.e. partition the set of such pairs into two subsets), there will be infinitely many numbers all pairs of which have the same color [20]. This theorem, which we denote RT^2_2, has been significant in the history of reverse mathematics [21, 18]. A "zoo" of other theorems [7] have been shown to fall outside the Big Five hierarchy at the same level, and an ongoing program of research is to study the structure of this zoo [15]. A first step in the analysis of this structure is to take the classical approach of reverse mathematics and to examine the theorems through the lens of provability over the base theory RCA_0. If one theorem suffices to prove another in this way, the first is stronger than the second; if the two are not equivalent, the first is *strictly* stronger.

But such an analysis does not reveal the whole picture. One finds clusters of theorems which on this account are equivalent—including clusters of equivalent theorems which have intuitively different strengths—

and more work is required to uncover, for example, the internal structure of these clusters. In this case one may wish to ask not only whether one theorem may be proved from another but *how* it may be proved. Since we are concerned with the strength of theorems, we might ask about the strength of proofs; that is, about the computable strength or complexity which is necessary to prove one of the theorems in our zoo from another such theorem, given the base theory RCA_0. In the last few years a model [9] has been developed to calibrate the strength required in these proofs. In this model one asks of each proof whether it can be carried out computably, and if so whether the computations can be strong [8], uniform [5], or both. Here *strong* refers to a certain technical restriction, to be defined hereafter, on the computation involved in a proof. This model has proven fruitful in unraveling the structure of the zoo; see for example [23, 13, 19, 1, 10, 4]. Moreover, this model has succeeded in some cases in capturing in formal terms the intuition that two logically equal theorems represent assertions of different strengths, a success which will be repeated hereafter. The computable relationships between theorems which are employed in this model are collectively referred to as *computable reductions*, *effective reductions*, or *strong reductions*. We will favor *strong reductions*.

This sets the stage on which the mathematics in this dissertation will play out. We select a particular menagerie of four theorems within the reverse mathematics zoo—to wit, certain stable versions of RT^2_2—and examine them under the lens of strong reductions. These theorems are naturally occurring in combinatorics and are logically equivalent to one another over the base theory RCA_0. Each theorem asserts the existence of an infinite set with certain properties, and because these properties seem to describe sets of different complexity, intuitively the theorems which assert the existence of more complex sets are stronger than those which assert the existence of less complex sets. This intuitive complexity can be expressed in terms of how difficult it would be to construct the set, i.e. can be expressed in terms of effectivity and constructive proof. As a result, the framework of strong reductions aptly formalizes the intuitive distinctions between the inhabitants of our menagerie. Hereafter we will explore which of the strong reduction relationships hold and fail to hold between the four theorems and eventually give a complete account of the structure of the menagerie.

1.2 Our Menagerie

We study four combinatorial principles related to Ramsey's theorem for pairs. Let us begin with the requisite definitions.

Definition 1.1.

- If $S \subseteq \omega$, then $[S]^2$ denotes the set of all 2-element subsets of S.
- A *2-coloring of pairs* is a function $f : [\omega]^2 \to 2$. We hereafter write $f(x, y)$ for $f(\{x, y\})$ with $x < y$, and "coloring" should be understood to mean "2-coloring of pairs."
- A 2-coloring of pairs is *stable* if $\lim_u f(x, u)$ exists for all $x \in \omega$.

Given a coloring, we are interested in the existence of sets homogeneous for the coloring in the following senses.

Definition 1.2.

- An infinite set $H \subseteq \omega$ is *homogeneous* for a coloring f if $f \upharpoonright [H]^2$ is constant.
- An infinite set $H = H_L \oplus H_R$ is *polarized homogeneous* or p-*homogeneous* for a coloring f if $f \upharpoonright H_L \times H_R$ is constant, and is *increasing* p-*homogeneous* if $f \upharpoonright \{\{x, y\} : x < y,\ x \in H_L,\ y \in H_R\}$ is constant.
- An infinite set H is *limit homogeneous* for a coloring f if $g(x) =_{\text{def}} \lim_u f(x, u)$ is total and constant on H.

With these definitions in hand, we can now introduce the menagerie of combinatorial principles in whose relative strength we are interested. All four principles are stable variants of RT^2_2. In fact the first of our principles, which has been the subject of considerable work in reverse mathematics (see for example [2, 6, 17]), is the stable Ramsey's theorem for pairs, SRT^2_2.

Statement 1.3 (SRT^2_2: Stable Ramsey's Theorem for pairs)**.** For every stable 2-coloring of pairs $f : [\omega]^2 \to 2$ there exists a set H homogeneous for f.

The next two principles were first studied in the context of combinatorics by Erdős and Rado in [14] and in the context of reverse mathematics by Dzhafarov and Hirst in [11].

Statement 1.4 (SPT^2_2: Stable Polarized Ramsey's Theorem for pairs)**.** For every stable 2-coloring of pairs $f : [\omega]^2 \to 2$ there exists a set H p-homogeneous for f.

Statement 1.5 (SIPT^2_2: Stable Increasing Polarized Ramsey's Theorem for pairs)**.** For every stable 2-coloring of pairs $f : [\omega]^2 \to 2$ there exists a set H increasing p-homogeneous for f.

As its conventional abbreviation suggests, the fourth principle with which we are concerned has a different lineage from the other stable variants of Ramsey's theorem. Rather than originating in combinatorics, the limit Ramsey's theorem for pairs D^2_2 originates in computability theory and reverse mathematics [2, 23], where its original statement had to do with infinite subsets of Δ^0_2 sets and their complements. Since a Δ^0_2 approximation can be construed as a 2-coloring of pairs, the principle in an equivalent form is a variant of Ramsey's theorem for pairs.

Statement 1.6 (D^2_2: Limit Ramsey's Theorem for pairs)**.** For every stable 2-coloring of pairs $f : [\omega]^2 \to 2$ there exists a set H limit homogeneous for f.

While the proof is highly nontrivial, Chong, Lempp, and Yang have shown [3] that each of these four combinatorial principles is equivalent to the others (and weaker than RT^2_2) over RCA_0: $\mathsf{RCA}_0 \vdash \mathsf{D}^2_2 \leftrightarrow \mathsf{SIPT}^2_2 \leftrightarrow \mathsf{SPT}^2_2 \leftrightarrow \mathsf{SRT}^2_2 \not\to \mathsf{RT}^2_2$. Nevertheless, if we think of what it would take to construct the different kinds of homogeneous set guaranteed by these principles, it would appear that SRT^2_2 makes a constructively stronger claim than SPT^2_2, which makes a constructively stronger claim than SIPT^2_2, which makes a constructively stronger claim than D^2_2. In other words, there should be some effective notion of relative strength and some corresponding relation $<_{\text{effective}}$ such that

$$\mathsf{D}^2_2 <_{\text{effective}} \mathsf{SIPT}^2_2 <_{\text{effective}} \mathsf{SPT}^2_2 <_{\text{effective}} \mathsf{SRT}^2_2.$$

Hereafter we will define such notions precisely and prove that strong reductions can recapture this intuition about the relative strengths of these principles.

1.3 Π^1_2 Principles and Strong Reductions

The stable versions of Ramsey's theorem for pairs with which we are concerned, together with many of the other theorems studied by reverse mathematics, are set existence theorems which are naturally stated in a Π^1_2 syntactic form, i.e. in the form

$$\forall X(\Phi(X) \to \exists Y \Psi(X, Y)),$$

where Φ and Ψ are arithmetical predicates. In our case $\Phi(X)$ would translate to "X is a stable 2-coloring of pairs" in English while $\Psi(X, Y)$ would translate to "Y is an infinite set homogeneous (in some particular sense) for X." It is helpful for our purposes to think of principles of this form as *problems*. Given a problem P expressed as in the formula displayed above, we call objects X such that $\Phi(X)$ holds *instances* of the problem and objects Y such that $\Psi(X, Y)$ holds *solutions* to the instance X of the problem. Given this language, we define as in [9] the following notions of effective reducibility between two problems.

Definition 1.7. Let P and Q be problems.

- P is *computably reducible* to Q, written $\mathsf{P} \leq_{\mathrm{c}} \mathsf{Q}$, if every instance X of P computes an instance $\widehat{X}$ of Q such that whenever $\widehat{Y}$ solves $\widehat{X}$, $X \oplus \widehat{Y}$ computes a solution Y to X.

- P is *strongly computably reducible* to Q, written $\mathsf{P} \leq_{\mathrm{sc}} \mathsf{Q}$, if every instance X of P computes an instance $\widehat{X}$ of Q such that whenever $\widehat{Y}$ solves $\widehat{X}$, $\widehat{Y}$ computes a solution Y to X.

- P is *Weihrauch reducible* to Q, written $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$, if there are Turing functionals Φ and Γ such that whenever X is an instance of P, Φ^X is an instance of Q, and whenever $\widehat{Y}$ solves Φ^X, $\Gamma^{X \oplus \widehat{Y}}$ solves X.

- P is *strongly Weihrauch reducible* to Q, written $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$, if there are Turing functionals Φ and Γ such that whenever X is an instance of P, Φ^X is an instance of Q, and whenever $\widehat{Y}$ solves Φ^X, $\Gamma^{\widehat{Y}}$ solves X.

Each of these notions is meant to capture the intuitive idea that if one has the ability to solve Q, one may in an algorithmic way use this ability to solve P. While such ideas have been used at least implicitly in the reverse mathematics literature for some time, they were first presented as objects of study quite recently. Weihrauch and strong Weihrauch reducibility were introduced by Weihrauch in [24] in the context of degrees of discontinuity, and in the context of reverse mathematics were independently discovered by

$$\begin{tikzcd}
& \leq_{\mathrm{sW}} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\leq_{\mathrm{sc}} \arrow[dr, Rightarrow] & & \leq_{\mathrm{W}} \arrow[dl, Rightarrow] \\
& \leq_{\mathrm{c}} &
\end{tikzcd}$$

Figure 1. Implications between types of strong reduction. The double arrows indicate implication, e.g. if $P \leq_{\mathrm{sW}} Q$ then $P \leq_{\mathrm{W}} Q$.

Dorais et al. [5]. Computable reducibility was developed by Dzhafarov [8]. These four notions of effective reducibility are related as in Figure 1. No other implications hold [16].

1.4 The Structure of the Menagerie

In this dissertation we will determine which of the strong reductions introduced in Definition 1.7 hold and do not hold between the four theorems introduced in Statements 1.3–1.6. Because a set H which is homogeneous for a stable coloring f is also limit homogeneous for f; and because it is easy to partition H into two infinite subsets so that their join $H' = H_L \oplus H_R$ is p-homogeneous for f; and because a set p-homogeneous for f is also increasing p-homogeneous for f, it is easy to see that $\mathsf{D}^2_2 \leq_{\mathrm{sW}} \mathsf{SIPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SRT}^2_2$. So far this agrees with our intuitions about the relative strengths of these four principles. But in order to fully capture this intuition, at least one type of reduction must fail in the other direction. This is a common thread in the use of strong reductions in reverse mathematics: not necessarily every notion of reducibility will separate—i.e. hold in only one direction between—two theorems, and we must find out which reduction(s) capture the differences in complexity between the principles. Indeed, the theorems do not separate under computable reducibility: $\mathsf{D}^2_2 \equiv_{\mathrm{c}} \mathsf{SIPT}^2_2 \equiv_{\mathrm{c}} \mathsf{SPT}^2_2 \equiv_{\mathrm{c}} \mathsf{SRT}^2_2$. (That $\mathsf{D}^2_2 \equiv_{\mathrm{c}} \mathsf{SRT}^2_2$ is proven as Lemma 3.5 in [2]. The remaining equivalences follow from the strong Weihrauch reductions indicated above together with the following result.) Weihrauch reduction fails to separate three of the theorems, as we now prove.

Proposition 1.8. $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} \mathsf{SPT}^2_2 \leq_{\mathrm{W}} \mathsf{SIPT}^2_2$.

We will use the following lemma to prove Proposition 1.8.

Lemma 1.9. Fix $i < 2$. There is a Turing functional Φ such that if $f : [\omega]^2 \to 2$ is a stable 2-coloring of pairs and L is an infinite set limit homogeneous for f with color i, then $\Phi^{f \oplus L \oplus \{i\}}$ describes an infinite set H homogeneous for f.

Proof of Lemma 1.9. Fix f and L. We may compute from $f \oplus L \oplus \{i\}$ an infinite set $H = \{h_0, h_1, h_2, \dots\}$ homogeneous for f as follows. Let h_0 be the least element of L, h_1 the least element of L such that $\{h_0, h_1\}$ is finite homogeneous for f with color i, and h_{k+1} the least element of L such that $\{h_0, h_1, \dots, h_k\}$ is finite homogeneous for f with color i. The stability of f and the fact that L is limit homogeneous for f with color i imply that this enumeration is computable from f, L, and i; this together with the ordering of enumeration implies that H is computable in the given data. $\square$

Proof of Proposition 1.8. Fix a stable 2-coloring of pairs $f : [\omega]^2 \to 2$ and an infinite set $I = I_0 \oplus I_1$ increasing p-homogeneous for f. Let i_0 be the least element of I_0 and i_1 the least element of I_1 such that $i_0 < i_1$. Because f is stable and I is increasing p-homogeneous for f, I_0 is limit homogeneous for f with color $f(i_0, i_1)$. Since $f(i_0, i_1)$ and I_0 are computable from $f \oplus I$, by the lemma $f \oplus I$ computes an infinite set H homogeneous for f, hence the infinite set $H \oplus H$ p-homogeneous for f. $\square$

By contrast to Proposition 1.8, Dzhafarov [9] showed that $\mathsf{SRT}^2_2 \not\leq_{\mathrm{W}} \mathsf{D}^2_2$ and $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{D}^2_2$. We strengthen the latter result by replacing SRT^2_2 with SIPT^2_2 as follows.

Theorem 1.10. $\mathsf{SIPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{D}^2_2$.

This follows by a straightforward adaptation of the proof of Corollary 3.6 in [9], but in addition to that proof we provide the following much simpler one.

Proof of Theorem 1.10. We will choose $f : [\omega]^2 \to 2$ be a non-computable instance of SIPT^2_2 all of whose solutions compute $\emptyset'$. To that end, fix a c.e. approximation of $\emptyset'$, $\{X_s\}_{s\in\omega}$, with least modulus μ. Now define

$$f(x, y) = \begin{cases} 0, & \text{if } y - x \leq \max\{\mu(z) : z \leq x\}; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\lim_u f(x, u) = 1$ for all $x \in \omega$. Let $H = H_L \oplus H_R$ be any solution to f (so H is increasing p-homogeneous for f with color 1). Then if $z \in \omega$, we can compute from H whether $z \in \emptyset'$ as follows: find the least $x \in H_L$ with $z \leq x$, and the least $y > x$ with $y \in H_R$. Since $f(x, y) = 1$, $y - x > \mu(z)$. Thus $z \in \emptyset'$ if and only if $z \in X_{y-x}$.

So every solution to f computes $\emptyset'$. But every instance of D^2_2—in particular, every instance computable from f—has a solution which does not compute $\emptyset'$ (Theorem 3.5, [12]), hence does not compute any infinite set increasing p-homogeneous for f. $\square$

In order to complete the task of mapping out all of the strong reductions which hold or do not hold between the four theorems in our menagerie, it remains to settle whether SRT^2_2, SPT^2_2 and SIPT^2_2 are equivalent under strong Weihrauch or strong computable reduction. In fact they are not, and hereafter we will prove the following two theorems.

Theorem 1.11 (First Main Theorem)**.** $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SPT}^2_2$.

Theorem 1.12 (Second Main Theorem). $\mathsf{SPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SIPT}^2_2$.

Thus it is the *strong* computable or Weihrauch definitions of reducibility which suffice to separate the theorems in our menagerie and to capture our intuitions about their relative strength: $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SIPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{D}^2_2$ and $\mathsf{D}^2_2 \leq_{\mathrm{sW}} \mathsf{SIPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SRT}^2_2$.

2 Tree Labeling and Forcing

In this section we will outline and formalize some techniques which will be used in the proofs of both of the main theorems.

2.1 Tree Labeling

To prove the main theorems we will use a pair of *tree labeling* arguments. The tree labeling method was first introduced by Dzhafarov in [9] to prove that $\mathsf{COH} \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_2$, and has since been used by Dzhafarov, Patey, Solomon, and Westrick in [13] to prove that $\mathsf{COH} \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_{<\infty}$ and that, for $k > l$, $\mathsf{RT}^1_k \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_l$.

The idea behind this proof technique is as follows. Suppose we have a Turing functional Φ, a finite set F, and an infinite reservoir $R \subseteq \omega$ such that $F < R$ (meaning that every element of R is greater than every element of F), and suppose we wish to extend F to $F \cup F_0$ where $F_0 \subseteq R$ is a finite set of elements taken from R with property that $\Phi^{F \cup F_0}(a) \downarrow = 1$ while $\Phi^F(a) \uparrow$. In other words, we wish to extend the finite set F to a larger finite set, drawing from the given reservoir, such that F so extended is oracle enough for Φ to converge on one new input. In the tree labeling construction, we consider the branching process of extending F by any one element from R, then by one more element from R, and so on. We represent these possible ways of extending F by adding different elements of R as an infinitely branching tree. The rule by which we produce this tree—call it T—is that $\emptyset \in T$ and that a nonempty string $\alpha \in R^{<\omega}$ belongs to T just in case α is increasing and there is no finite set $F_0 \subseteq \mathsf{ran}\,(\alpha \restriction |\alpha| - 1)$ with the property that $\Phi^{F \cup F_0}(a) \downarrow = 1$ while $\Phi^F(a) \uparrow$. Notice that the preceding property is asserted for F_0 contained in the range not of α but of α minus its last element. In this way we ensure that any well-founded path p through T will terminate precisely when there is $F_0 \subseteq \mathsf{ran}(p)$ such that $\Phi^{F \cup F_0}$ converges on one new input. We introduce a labeling scheme by which every terminal node of the tree bears a *label*, namely that one new input (or the least such, in case there is more than one way to choose $F_0 \subseteq \mathsf{ran}(p)$ each leading to convergence on a different new input). This labeling scheme then propagates these labels recursively back down the tree from leaf to root.

A tree labeling argument thus reduces the process of searching for a way to extend a finite set to produce convergence of a functional on one new element to a process of searching through an infinitely branching tree. If we require additional constraints to hold about the way we extend the given finite set, this translates to additional constraints on the way we search for a suitable branch of the tree. The labels attached to the nodes of the tree will usefully guide such searches.

2.2 Formalizing a New Approach

The forgoing outline describes the tree labeling method used in earlier papers. The proofs here will require elaborations on previous methods. It is possible to alter the technique such that the extensions of the given finite set lead not to just one new convergence but to any desired finite number of new convergences. Our proofs of the main theorems will require, depending on the case, convergence on two or three new inputs so that these new inputs can be used in the more complicated diagonalization requirements of these proofs. This requirement necessitates, among other changes, that we replace the labeling scheme used in previous tree labeling arguments with a new one suitable to any finite number of labels. For simplicity, we will define the construction for a tree with 2 labels; the definitions required to construct a tree with 3 labels will then be clear, as the definitions for the 2-label case will apply *mutatis mutandis*. In that regard, we call the reader's attention particularly to Remark 2.8. Additional changes to the construction will be made to account for the fact that we wish to use this method specifically to construct p-homogeneous sets. In order to simplify the exposition, we begin with the following definitions.

Definition 2.1.

- If α is a nonempty string of natural numbers, then $\alpha^{\#}$ denotes $\alpha \upharpoonright |\alpha| - 1$.
- If α and β are strings of natural numbers, then $\alpha * \beta$ denotes α concatenated by β.
- If α is a string of natural numbers and $x \in \omega$, then $\alpha * x$ denotes $\alpha * \langle x \rangle$.
- If $\alpha, \alpha * x \in T$ for some tree T, we say that $\alpha * x$ is a *successor* of α in T.
- If $A, B \subseteq \omega$, as above we write $A < B$ if every element of A is less than every element of B.

We now proceed to define *tree labeling with two labels*. The choice of letter names is somewhat prescient: when we apply this method of proof, many of the same letter names will appear.

Definition 2.2. Let $H \subset \omega$ be finite and $I \subset \omega$ be infinite, with $H < I$. Let Γ be a Turing functional. Finally, let $k \in \omega$. We define the tree $T(k, \Gamma, H, I) \subseteq I^{<\omega}$ by $\emptyset \in T(k, \Gamma, H, I)$ and for a nonempty string α,

$\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \mathsf{ran}(\alpha^{\#})$ and no $b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = 1.$$

In the tree labeling arguments we later make, H will represent an initial segment of a p-homogeneous set that has been built by some stage of the construction, while F_L and F_R will represent possible additions to be made to the left and right columns, respectively, of that set. Then the tree $T(k, \Gamma, H, I)$ will represent, as explained above, the branching possible ways by which we can extend H to $H \cup (F_L \oplus F_R)$. The number k will be chosen so that a and b are large enough to be unencumbered by certain commitments we will have made in the forcing conditions in our construction.

Remark 2.3. $T = T(k, \Gamma, H, I)$ has the following three properties:

- If T is not well-founded and p is any infinite path through T, then $\mathsf{ran}(p) \subseteq I$ is infinite and for all $F_L, F_R \subseteq \mathsf{ran}(p)$ and all $b > a \geq k$,

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \simeq 0 \text{ or } \Gamma^{H \cup (F_L \oplus F_R)}(b) \simeq 0.$$

- If $\alpha \in T$, then if α is not terminal and $\mathsf{ran}(\alpha) < x \in I$, $\alpha * x \in T$.
- If $\alpha \in T$ is terminal, then

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = 1$$

  for some $F_L, F_R \subseteq \mathsf{ran}(\alpha)$ and some $b > a \geq k$.

We now define the labeling scheme by which we record at each node of the tree the information we need to search through the tree in the proofs of the main theorems.

Definition 2.4. When $T = T(k, \Gamma, H, I)$ is well-founded, we *label* the nodes of T recursively, starting at the terminal nodes. Each node is labeled with an ordered pair whose elements may be natural numbers or the symbol ∞.

- If $\alpha \in T$ is terminal, we label α with the least (in the lexical sense) pair $\langle a, b \rangle$ of elements $b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = 1$$

for some $F_L, F_R \subseteq \mathsf{ran}(\alpha)$.

- If $\alpha \in T$ is not terminal, then we determine the label of α starting with the second element as follows:
  - If there is any $b \in \omega$ such that infinitely many of the successors of α have labels with second element b, then we let the least such b be the second element of the label of α. Otherwise, we let ∞ be the second element of the label of α.
  - Now suppose the second element b of the label of α has been determined already; we will determine its first element a according as $b \in \omega$ or $b = \infty$. If $b = \infty$, then we let a be the least finite number appearing as the first element of the label of infinitely many successors of α, or else if there is no such finite number we let $a = \infty$. If $b \in \omega$, we restrict our attention to just those successors of α whose labels' second element is b, and let a be the least number appearing as the first element of the label of infinitely many successors of α. Observe that $a < b$.

Remark 2.5. Note that in this labeling scheme no label which has the symbol ∞ as its first element has a finite number as its second.

Definition 2.6. Suppose $T = T(k, \Gamma, H, I)$ is well-founded. Then the *labeled subtree* $T^L = T^L(k, \Gamma, H, I)$ of T is obtained from T as follows. First, the root node of T (namely $\emptyset$) is added to T^L. Now suppose we have added to T^L some non-terminal node α of T. We then add to T^L some of the successors of α, thus:

- If α has label $\langle a, b\rangle \in \omega^2$ in T, then we add to T^L all those successors of α with the same label.
- If α has label $\langle a, \infty\rangle$ for some $a \in \omega$ in T, then if infinitely many successors of α have label $\langle a, b\rangle \in \omega^2$, then for each $b \in \omega$ such that $\langle a, b\rangle$ appears as the label of a successor of α, we select the least x such that $\alpha * x$ has that label and add $\alpha * x$ to T^L; and if on the other hand cofinitely many successors of α have the same label (namely $\langle a, \infty\rangle$), we add all such successors to T^L.
- Otherwise, if α has label $\langle \infty, \infty\rangle$, then if infinitely many successors of α have label $\langle a, b\rangle \in \omega^2$, then for each such pair that appears as the label of a successor of α, we select the least x such that $\alpha * x$ has that label and add $\alpha * x$ to T^L; and if cofinitely many successors of α have label $\langle a, \infty\rangle$ for $a \in \omega$, then for each $a \in \omega$ such that $\langle a, \infty\rangle$ appears as the label of a successor of α, we select the least x such that $\alpha * x$ has that label and add $\alpha * x$ to T^L; and otherwise if cofinitely many successors of α have label $\langle \infty, \infty\rangle$, we add those successors to T^L.

All the nodes of T^L retain the labels they had as nodes of T. Note that every node terminal in T is also terminal in T^L, and that every non-terminal node in both T and T^L has infinitely many successors.

Definition 2.7. A node $\alpha \in T^L$ is called a *transition node* if the symbol ∞ appears in the label of α, and appears strictly fewer times and no more than once in the label of each successor of α.

Remark 2.8 (Tailoring Definition 2.7 for the case of 3 labels.)**.** The use of the words "and no more than once" in the preceding definition may appear peculiar. This way of stating the definition is necessary in order to live up to our earlier promise that every definition for tree labeling with 2 labels would adapt straightforwardly to tree labeling with 3 labels. By wording the definition of *transition node* in this way, we are able to use the same definition for both the 2-label and the 3-label case.

2.3 A Notion of Forcing

In the proofs that follow we will use the following notion of forcing.

Definition 2.9. Let $\mathbb{C}$ denote the following notion of forcing. A condition is an ordered triple $p = \langle \sigma^p, l^p, |p| \rangle$ where $|p| \in \omega$, $\sigma^p : [|p|]^2 \to 2$, $l^p : |p| \to 2 \times \omega$, and $l^p(x) = \langle i, z \rangle$ implies that if $\sigma^p(x, y)$ is defined and $y \geq z$ then $\sigma^p(x, y) = i$.

From any sufficiently generic filter $\mathcal{G}$ for $\mathbb{C}$ we obtain a stable 2-coloring of pairs $f = \bigcup_{p \in \mathcal{G}} \sigma^p : [\omega]^2 \to 2$ together with a function $l = \bigcup_{p \in \mathcal{G}} l^p : \omega \to 2 \times \omega$ such that for each $x \in \omega$, $\lim_u f(x, u) = (l(x))_0$. Thus $|p|$ is an initial segment of ω on which we define a partial 2-coloring of pairs σ^p together with some commitments l^p about σ^p intended to ensure that we build a *stable* 2-coloring of pairs. That is, $l^p(x) = \langle i, z \rangle$ is intended to represent the commitment that $\sigma^p(x, y)$ will stabilize to color $i \in 2$ for $y \geq z$.

We have need of the following genericity lemma.

Lemma 2.10. If $p_0 \geq p_1 \geq p_2 \geq \cdots$ is a sequence of $\mathbb{C}$-conditions which is 3-generic relative to some set $P \subseteq \omega$, and if $f = \bigcup_s \sigma^{p_s} : [\omega]^2 \to 2$, then $f \oplus P$ does not compute a homogeneous or a p-homogeneous set for f.

Proof. First observe that if $f \oplus P$ were to compute a p-homogeneous set for f, then from that set together with f one could compute a homogeneous set for f. Hence we need only prove that under the given hypotheses $f \oplus P$ does not compute a homogeneous set for f.

Fix $P \subseteq \omega$ such that $p_0 \geq p_1 \geq p_2 \geq \cdots$ is 3-generic with respect to P and fix a Turing functional Γ. Let $W_{P,\Gamma}$ be the set of all conditions p which force one of the following two statements:

(a) $\Gamma^{f \oplus P}$ does not define an infinite set;

(b) there are $x, y \in \omega$ with $\Gamma^{f \oplus P}(x) \downarrow = \Gamma^{f \oplus P}(y) \downarrow = 1$ and $\lim_u f(x, u) \neq \lim_u f(y, u)$.

$W_{P,\Gamma}$ is Σ^0_3-definable in P, and we claim that $W_{P,\Gamma}$ is dense in $\mathbb{C}$. To see this, let p be any condition none of whose extensions force (a), and suppose by way of contradiction that no extension of p forces (b). This means that for every $x, y \in \omega$, if $q \leq p$ then q does not force both that $\lim_u f(x, u) \neq \lim_u f(y, u)$ and that $\Gamma^{f \oplus P}(x) \downarrow = \Gamma^{f \oplus P}(y) \downarrow = 1$. Then in particular there are no $x, y \geq |p|$, τ extending σ^p, and L extending l^p such that

(1) τ respects L;

(2) $\Gamma^{\tau \oplus P \upharpoonright |p|}(x) \downarrow = \Gamma^{\tau \oplus P \upharpoonright |p|}(y) \downarrow = 1$; and

(3) $(L(x))_0 \neq (L(y))_0$.

Now if there is no such x, y, τ, L satisfying (1) and (2), then p forces that $\Gamma^{f \oplus P}$ does not define an infinite set, which is a contradiction. Therefore there are x, y, τ, L satisfying (1) and (2). But (3) is independent of (1) and (2); given x, y, τ satisfying (1) and (2) we may find L extending l^p and compatible with τ such that $(L(x))_0 \neq (L(y))_0$. We conclude that W is dense in $\mathbb{C}$. $\square$

3 Separating SRT^2_2 and SPT^2_2

In this section we will prove the following theorem, which has the First Main Theorem as its corollary.

Theorem 3.1. There exists a stable 2-coloring of pairs $f : [\omega]^2 \to 2$ and a family Y of infinite sets such that no $(f \oplus P)$-computable set is homogeneous for f for any $P \in Y$, and every stable 2-coloring of pairs $f' : [\omega]^2 \to 2$ computable from f has either an $(f \oplus P)$-computable p-homogeneous set for some $P \in Y$, or if not then some p-homogeneous set which does not compute a set homogeneous for f.

Before proving Theorem 3.1, we observe how the First Main Theorem is a direct consequence. Let f and Y be as in the statement of Theorem 3.1, let Φ, Ψ be any Turing functionals, and suppose that Φ^f is a stable 2-coloring of pairs. Let H be set p-homogeneous for Φ^f. Then by Theorem 3.1, Ψ^H is not homogeneous for f. Thus $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SPT}^2_2$.

Proof of Theorem 3.1. We build

- a sequence of $\mathbb{C}$-conditions $p_0 \geq p_1 \geq p_2 \geq \cdots$ with $\lim_s |p_s| = \infty$;
- sequences of finite sets (initial segments of p-homogeneous sets) $H^\Phi_{j,0} \subseteq H^\Phi_{j,1} \subseteq H^\Phi_{j,2} \subseteq \cdots$ for each Turing functional Φ and each $j < 2$;
- a sequence of infinite sets (reservoirs) $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $H^\Phi_{j,s} < I_s$ for each Φ, j, s;
- a sequence of finite families $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$ of infinite subsets of ω;

and we define $f = \bigcup_s \sigma^{p_s}$, $H^\Phi_j = \bigcup_s H^\Phi_{j,s}$ for each $j < 2$, and $Y = \bigcup_s Y_s$. The construction will ensure the following requirements, for all $i \in \omega$ and Turing functionals Φ, Γ, Δ:

$\mathcal{P}_i$: the sequence $p_0 \geq p_1 \geq p_2 \geq \cdots$ is 3-generic relative to each $P \in Y_i$;

$\mathcal{Q}_{\Phi,i}$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$-computable p-homogeneous set for some $P \in Y$ or else both H^Φ_0 and H^Φ_1 are infinite in both columns (in other words, given our encoding of p-homogeneous sets, contain infinitely many even and infinitely many odd numbers);

$\mathcal{R}^{\Phi}_{\Gamma,\Delta}$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$-computable p-homogeneous set for some $P \in Y$; or else if $\Gamma^{H_0^\Phi}$ defines an infinite set then this set is not homogeneous for f; or else if $\Delta^{H_1^\Phi}$ defines an infinite set then this set is not homogeneous for f.

By way of explaining the $\mathcal{Q}$ and $\mathcal{R}$ requirements, note that by Lemma 2.10, if there is any sequence satisfying the $\mathcal{P}$ requirements and such that for some $P \in Y$ $f \oplus P$ computes a set p-homogeneous for Φ^f, there will then be a set which is p-homogeneous for Φ^f but which computes no set homogeneous for f.

3.1 Outline of the Construction

We are about to begin a very involved forcing argument. Cases and subcases will multiply, and so it will be helpful to first consider an informal outline of the process. We are setting out to prove that $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SPT}^2_2$ and while per the statement of Theorem 3.1 the technical details are somewhat more complicated, in essence we wish to produce a stable 2-coloring of pairs f and for each Turing functional Φ such that Φ^f is also a stable 2-coloring of pairs a set H that is p-homogeneous for Φ^f but which does not compute any set homogeneous for f. In the language of problems explained in the introduction, our goal is to produce an instance f of SRT^2_2 such that any instance Φ^f of SPT^2_2 has a solution H which does not compute any solution to f. To accomplish this in stages, we extend f over longer and longer initial segments of ω while, at the same time and for each Φ, enlarging the finite approximation of H (more precisely, H_0 for color 0 and H_1 for color 1) by successive finite extensions. Really we are performing two forcing constructions in parallel: Cohen forcing for the coloring and—though we will not dress it up as such—Mathias forcing via tree labeling for the p-homogeneous sets.

The work of finding the correct extensions for successive finite approximations to the p-homogeneous sets (which appears in the proof as the work to satisfy the $\mathcal{R}$ requirements) will be accomplished using tree labeling constructions. Assuming for the purpose of this outline that the labeled subtree at a given stage is well-founded, we will search for a branch of the tree with the property that its range contains an appropriate finite set by which to extend the approximation of the p-homogeneous set under consideration. But there is an important twist here which has been absent in previous tree labeling proofs: the search through the original labeled subtree may fail to find an appropriate extension for one of the p-homogeneous sets at the given stage, but when it does so the information gained from that very failure ensures that a new, more elaborate tree labeling construction must succeed in extending the p-homogeneous set of the other color at the given stage. It is here that we leverage the combinatorial differences between homogeneous and p-homogeneous sets, and doing so will require us to label the trees with 3 labels rather than 2 as in the simpler case.

This and other concerns result in a branching logical structure to the proof, where cases beget subcases and subsubcases. There will be a Case I for when the first tree labeling construction succeeds and a Case II for when the first tree labeling construction fails and another is required. Then, under Case I or Case II, there are different concerns depending on whether we are, at a given step of the search through the labeled subtree, looking at a transition node or a non-transition node, as defined in Definition 2.7. The work to be done at non-transition nodes is straightforward, while at transition nodes the work divides further into cases depending on precisely which kind of transition node occurs. In one of these transition node cases, there will be an additional division into subcases based on some technical details of the tree labeling scheme. Because of the branching logical structure of the proof the mathematical notation will necessarily become somewhat complicated. In order to improve its readability, therefore, we abide by the following conventions. Careful attention to these conventions will allow the reader to keep track of the various moving parts of the construction.

Remark 3.2 (Notational Conventions)**.**

- We will use lowercase Greek letters to denote strings in the labeled subtree in any given tree labeling construction.
- We will use the letters a, b, and where necessary c to denote the components of labels in the labeled subtree, and distinguish between the same component from another label using subscripts or other notations. Thus a_i and a_j would be two numbers each of which shows up as the first component in the label of some string in the labeled subtree. Additionally, we will always have $a < b < c$.
- We will use the letters x, y, and z to denote elements in the range of strings in the labeled subtree, and specifically to denote elements by which a string may be extended to one of its successors in the labeled subtree. Thus if α is a string in the labeled subtree, we may write $\alpha * x$ to denote a successor of α in the labeled subtree. Additionally, we will always have $x < y < z$.

- Sometimes we will need to talk about the labels of the successors of a string α in the labeled subtree at the same time as we talk about the elements x by which α may be extended to its successors. At those times it will be convenient to associate the label $\langle a, b\rangle$ of $\alpha * x$ with the element x and to record this relationship by writing the label of $\alpha * x$ as $\langle a_x, b_x\rangle$. In those cases where we are using 3 labels instead of two, we will similarly write the label of $\alpha * x$ as $\langle a_x, b_x, c_x\rangle$.

- When, following the conventions for letter names described above, we quantify over x (or a, etc.), and wish at the same time to speak about a particular value of x, we will affix an asterisk to the letter name. Thus we may for example search among all the successors $\alpha * x$ of α in the labeled subtree to find a particular successor $\alpha * x^*$, or fix a number a^* and look at all possible numbers b such that $\langle a^*, b\rangle$ appears as the label of a string in some subset of the labeled subtree. In every case, a superscript asterisk denotes a particularized value, whether a fixed choice or the object of a search.

- We will use the letters p, q, and r to name forcing conditions.

- We will write a set formed as the join of two other sets as $H = H_L \oplus H_R$. We will refer to H_L as the *left* and H_R as the *right column* of H.

- When building a homogeneous or p-homogeneous set, we will record the color $i \in 2$ in the subscript. Thus H_i is a homogeneous or p-homogeneous set of color i. In the context of tuples, the subscripts 0 and 1 will be used in the usual way. That is, the function $(t)_0$ returns the 0^{th} component of t and the function $(t)_1$ returns the 1^{st} component of t.

3.2 Construction

Devote infinitely many stages $s \in \omega$ to each requirement. Let p_0 be any condition with $|p_0| = 0$. For each Φ let $H^{\Phi}_{0,0} = H^{\Phi}_{1,0} = \emptyset$, and let $I_0 = \omega$ and $Y_0 = \emptyset$. At stage $s+1$ assume by way of induction that we have p_s, $H^{\Phi}_{j,s}$ for $j < 2$ and all Φ, I_s, and Y_s and assume that if $H^{\Phi}_{j,s}$ is nonempty for some j and Φ, then p_s forces that Φ^f is a stable coloring of pairs and that $\Phi^f(x, y) = j$ whenever $2x, 2y+1 \in H^{\Phi}_{j,s}$ or when $y \in I_s$ and either $2x \in H^{\Phi}_{j,s}$ or $2x+1 \in H^{\Phi}_{j,s}$. At the end of a stage any of p_{s+1}, $H^{\Phi}_{j,s+1}$, I_{s+1}, or Y_{s+1} not yet defined should be taken to be identical to p_s, $H^{\Phi}_{j,s}$, I_s, or Y_s, respectively.

$\mathcal{P}$ requirements

Suppose s is dedicated to requirement $\mathcal{P}_i$ for some $i < s$ and that it is the $\langle n, m\rangle^{\text{th}}$ such stage. If $n > |Y_i|$ do nothing. Otherwise, let P be the n^{th} member of the family Y_i in some fixed enumeration and let W be the m^{th} $\Sigma^0_3(P)$ set in some fixed enumeration. If p_s has an extension q in W, fix q and let $p_{s+1} = q$, so that $p_0 \geq p_1 \geq \cdots \geq p_s \geq p_{s+1} \geq \cdots$ meets W. Otherwise, do nothing, and $p_0 \geq p_1 \geq \cdots \geq p_s \geq p_{s+1} \geq \cdots$ avoids W.

$\mathcal{Q}$ requirements

Suppose s is dedicated to $\mathcal{Q}_{\Phi,i}$. Without loss of generality we assume that p_s decides whether or not Φ^f is a stable 2-coloring of pairs. If p_s forces that Φ^f is not such a coloring, do nothing. Otherwise, we consider two cases.

- If for some $j < 2$ and $k \in \omega$ there is no extension of p_s which forces that $\lim_u \Phi^f(x, u) = j$ for some $x \geq k$ in I_s, then $P = \{x \in I_s : x \geq k\}$ is limit homogeneous for Φ^f with color $1 - j$, and so $(f \oplus P)$ computes a set p-homogeneous for Φ^f. We set $Y_{s+1} = Y_s \cup \{P\}$. This satisfies the requirement (since, as remarked earlier, this means there is a set which is p-homogeneous for Φ^f but which computes no set homogeneous for f).
- If no such j, k exist, then there are numbers $x_{00}, x_{01}, x_{10}, x_{11} \in I_s$ and an extension of p_s forcing that $H^\Phi_{j,s} \cup \{2x_{j0}, 2x_{j1} + 1\}$ is finite p-homogeneous for Φ^f and $\lim_u \Phi^f(x_{ji}, u) = j$ for each $i, j < 2$. In this case let p_{s+1} be such an extension of p_s, let $H^\Phi_{j,s+1} = H^\Phi_{j,s} \cup \{2x_{j0}, 2x_{j1} + 1\}$, and let $I_{s+1} = \{x : m < x \in I_s\}$ where m is greater than the stabilization points under Φ^f of every element of $H^\Phi_{0,s+1} \cup H^\Phi_{1,s+1}$. Observe that both columns have been extended by one element.

$\mathcal{R}$ requirements

Suppose s is dedicated to $\mathcal{R}^\Phi_{\Gamma,\Delta}$ and assume that p_s forces that Φ^f is a stable coloring of pairs. The goal of this requirement is to extend by some finite set either the initial segment of H^Φ_0 (Case I) or the initial segment of H^Φ_1 (Case II) that we have constructed so far, subject to the following condition: if we extend H^Φ_0, then Γ does not compute a homogeneous set for f from any further extension of H^Φ_0; and if on the other hand we extend H_1^Φ, then Δ does not compute a homogeneous set for f from any further extension of H_1^Φ. When we have so extended one of H_0^Φ, H_1^Φ, we will say that we have successfully *diagonalized* against such computations.

We intend to accomplish this diagonalization in the following way. There will be two numbers, say a and b, and two finite sets F_L and F_R, all arising from a tree labeling construction, about which we know either that

$$\Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(b) \downarrow = 1$$

or else that

$$\Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(a) \downarrow = \Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(b) \downarrow = 1.$$

Let us suppose we know the first. In other words, we know that if we extend H_0^Φ by $F_L \oplus F_R$, then the set computed from $H_0^\Phi \cup (F_L \oplus F_R)$ by Γ will contain a and b. Thus to diagonalize—i.e. to ensure that the set so computed is not homogeneous for f—we will choose an extension q of the condition p_s such that $\sigma^q(a, b)$, $(l^q(a))_0$, and $(l^q(b))_0$ are not all equal. This will guarantee that no set containing both a and b is homogeneous for the coloring eventually obtained by extending σ^q. The tension of the proof arises from the fact that, while diagonalizing in this way, we also need q to force that elements of F_L and F_R have the right limits under Φ^f so that $H_0^\Phi \cup (F_L \oplus F_R)$ can in fact be extended to a p-homogeneous set.

Case I (Extending H_0^Φ)

Let $T_0 = T(|p_s|, \Gamma, H_{0,s}^\Phi, I_s)$. If T_0 is not well-founded then let I_{s+1} be the range of an infinite path through T_0. Observe that in this case the requirement is satisfied. If T_0 is well-founded, then let T_0^L be the labeled subtree of T_0.

We now try to define two sequences, conditions

$$p_s \geq q_0 \geq q_1 \geq q_2 \geq \cdots$$

and nodes of T_0^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \cdots$$

where for all $j \geq 0$, α_{j+1} is a successor of α_j and for all $j \geq 0$ the condition q_j forces that

$$\lim_u \Phi^f(x, u) = 0$$

for all $x \in \mathsf{ran}(\alpha_j)$. We begin the definition of these sequences as follows.

- If $\emptyset = \alpha_0$ has label $\langle a, b \rangle$ with $a, b \in \omega$, let q_0 be any extension of p_s having $\sigma^{q_0}(a, b)$, $(l^{q_0}(a))_0$, and $(l^{q_0}(b))_0$ not all equal. In this case the diagonalization for the present requirement is now complete.
- Otherwise, let $q_0 = p_s$.

We then proceed by induction until either the induction fails and we satisfy the requirement by adding a certain set P to Y, or else the induction always succeeds at every non-terminal node. In the latter case, once we reach a terminal node we will be ready to diagonalize. Suppose we have defined q_n and α_n and that the latter is not terminal in T_0^L. Recall that q_n forces that there is some $m \in \omega$ such that $\Phi^f(x, y) = 0$ for $x \in \mathsf{ran}(\alpha_n)$ and $y \geq m$. Let $S = S(n)$ be the set of all successors $\alpha_n * x$ of α_n with $x \geq m$. The induction breaks into cases according as α_n is or is not a transition node. At the first suitable transition node, we set up to diagonalize.

Case I.1 (Non-Transition Node)

If α_n is not a transition node, let $P = \{x : \alpha_n * x \in S\}$. We look for an $x^* \in P$ and an extension q of q_n which forces that

$$\lim_u \Phi^f(x^*, u) = 0.$$

If we find such, we let $q_{n+1} = q$ and we let α_{n+1} be any $\beta \in S$ with $\beta(n) = x^*$; if we find no such, then P is limit-homogeneous for Φ^f and thus $f \oplus P$ computes a p-homogeneous set for Φ^f (for example, we can thin P computably in f to a set G homogeneous for Φ^f; then $G \oplus G$ is p-homogeneous for Φ^f). In this case we set $Y_{s+1} = Y_s \cup \{P\}$ and $p_{s+1} = q_n$, satisfying the requirement and ending stage s.

Case I.2 (Transition Node)

If α_n is a transition node, the induction breaks into two cases. The header for each case gives a shorthand for the sort of transition being discussed. For example, where we write $\langle \infty, \infty \rangle \to \langle a, \infty \rangle$ we mean that the node α_n under consideration has label $\langle\infty,\infty\rangle$ while every successor of α_n has a label in which the symbol ∞ appears exactly once.

Case I.2.1 ($\langle\infty,\infty\rangle \to \langle a,\infty\rangle$)

If α_n has label $\langle\infty,\infty\rangle$ and every successor of α_n has a label in which the symbol ∞ appears exactly once, then we proceed as in the non-transition case.

Case I.2.2 ($\langle\infty,\infty\rangle \to \langle a,b\rangle$ or $\langle a,\infty\rangle \to \langle a,b\rangle$)

If α_n has a label in which the symbol ∞ appears but every successor of α_n is labeled only with finite numbers, then let

$$P = \{\langle x,a,b\rangle : \alpha_n * x \in S \text{ and } \alpha_n * x \text{ has label } \langle a,b\rangle \text{ and } b > |q_n|\}.$$

The idea behind the restriction that $b > |q_n|$ in the definition of P above is that we wish each b to be greater than all the numbers about which q_n has already made commitments. In particular, we wish to be free to determine for each b $(l^q(b))_0$ in an extension q of q_n. We look for a tuple $\langle x^*,a^*,b^*\rangle \in P$ and an extension q of q_n which forces that

$$\lim_u \Phi^f(x^*,u) = 0$$

and is such that $\sigma^q(a^*,b^*)$, $(l^q(a^*))_0$, and $(l^q(b^*))_0$ are not all equal. If we find such an extension q of q_n, we let $q_{n+1} = q$ and we let α_{n+1} be any $\beta \in S$ with $\beta(n) = x^*$ having label $\langle a^*,b^*\rangle$. Then, as α_{n+1} is not a transition node, we return to Case I.1. But it may be the case that there is no such extension. In that case we use information gleaned from the failure to find such an extension and use this information to refine the reservoir I_s, and we then jettison the tree T_0^L currently in use and switch from trying to extend H_0^Φ to trying to extend H_1^Φ.

We record the information gleaned from the failure case and update the reservoir as follows. Fix $Q \subseteq P$ to be a set of triples $\langle x,a_x,b_x\rangle$ such that $\{(x,b_x) : \langle x,a_x,b_x\rangle \in Q\}$ is a one-to-one function with infinite domain and range. A set Q matching these specifications must exist by the definition of the tree labeling scheme and the particular kind of transition node under consideration. We then redefine the reservoir I_s to be $\{x : \exists a_x, b_x\,(\langle x,a_x,b_x\rangle \in Q\}$. We are now finished with this subcase and will begin a new tree labeling construction in Case II.

Case II (Extending H_1^{Φ})

This is the case we are in after having already attempted to proceed as in Case I, but having failed as described above in Case I.2.2. Thus we begin the present case with considerable information, and this will allow us to perform a tree labeling construction similar to the one in Case I but with no possibility of failure. Recall that we have assumed that p_s (and hence q_n) forces that Φ^f is a stable 2-coloring of pairs, and recall that Case I.2.2 above failed because we could not find a tuple $\langle x^*, a^*, b^* \rangle \in P$ and an extension q of q_n which forces that $\lim_u \Phi^f(x^*, u) = 0$ and is such that $\sigma^q(a^*, b^*)$, $(l^q(a^*))_0$, and $(l^q(b^*))_0$ are not all equal. This means, given the definitions above of Q and the revised reservoir I_s, that for any $x \in I_s$ and any extension q of q_n such that $\sigma^q(a^*, b^*)$, $(l^q(a^*))_0$, and $(l^q(b^*))_0$ are not all equal q forces that $\lim_u \Phi^f(x^*, u) = 1$. In the tree labeling construction which follows, we will extend q_n several times. As we gradually extend the forcing condition, for any given x it might occur—and we may as well assume—that the limiting color of a_x will be forced, and that the stabilization point will fall below b_x so that the color of (a_x, b_x) will be forced also. However, since there are infinitely many different numbers b_x, by restricting our attention to those x's for which the limiting color of b_x has not yet been forced, we will always have the option of choosing an extension q as above, i.e. such that $\sigma^q(a_x, b_x)$, $(l^q(a_x))_0$, and $(l^q(b_x))_0$ are not all equal, and thus force that $\lim_u \Phi^f(x, u) = 1$. The fact that we can always accomplish this while making progress toward satisfying the present requirement underlies the remainder of the construction, and so we find it useful to define the following terms.

Definition 3.3. With Q and I_s as above, we define the function $\mathsf{button} : I_s \to \omega$ as follows. For any $x \in I_s$, there is a unique tuple $\langle x, a_x, b_x \rangle \in Q$. Let $\mathsf{button}(x) = b_x$.

Definition 3.4. With Q, I_s, and q_n as above and for any $\langle x, a_x, b_x \rangle \in Q$, a forcing extension q of q_n is said to *press the button of* x or to *press* $\mathsf{button}(x)$ if $\sigma^q(a_x, b_x)$, $(l^q(a_x))_0$, and $(l^q(b_x))_0$ are not all equal.

Thus any forcing extension which presses the button of x forces that $\lim_u \Phi^f(x, u) = 1$.

As promised, for this case we need to modify our definition of tree labeling. We will use *tree labeling with three labels*. The definition of $T(k, \Gamma, H, I)$ is for this case changed to the following: $\emptyset \in T(k, \Gamma, H, I)$ and for a nonempty string α, $\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \mathsf{ran}(\alpha^{\#})$

and no $c > b > a \geq k$ such that

$$\Gamma^{H\cup(F_L\oplus F_R)}(a)\downarrow=\Gamma^{H\cup(F_L\oplus F_R)}(b)\downarrow=\Gamma^{H\cup(F_L\oplus F_R)}(c)\downarrow=1.$$

The method for labeling the nodes of $T(k,\Gamma,H,I)$ extends the method from tree labeling with two labels in the natural way, as does the method for selecting the nodes of the labeled subtree $T^L(k,\Gamma,H,I)$. Remark 2.3 applies *mutatis mutandis*.

Now we begin a new tree labeling construction starting where we left off when Case I.2.2 failed. Thus we have the forcing condition $q_n \leq p_s$ and the revised reservoir I_s.

Remark 3.5. Our notation will be slightly more manageable if we allow ourselves to reuse the letter n rather than reserving n as the index of q_n, the forcing condition at which we ended Case I.2.2. Thus we will write q for that forcing condition hereafter.

Let $T_1 = T(|q|,\Delta,H^{\Phi}_{1,s},I_s)$. If T_1 is not well founded then let I_{s+1} be the range of an infinite path through T_1. Observe that in this case the requirement is satisfied. If T_1 is well founded, let T_1^L be the labeled subtree of T_1.

We now try to define two sequences, conditions

$$q \geq r_0 \geq r_1 \geq r_2 \geq \cdots$$

and nodes of T_1^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \cdots$$

where for all $j \geq 0$, α_{j+1} is a successor of α_j and for all $j \geq 0$ the condition r_j forces that

$$\lim_u \Phi^f(x,u) = 1$$

for all $x \in \mathsf{ran}(\alpha_j)$. We begin the definition of these sequences as follows.

- If $\emptyset = \alpha_0$ has label $\langle a,b,c\rangle$ with $a,b,c\in\omega$, let r_0 be any extension of q having $\sigma^{r_0}(a,b)$, $(l^{r_0}(a))_0$, and $(l^{r_0}(b))_0$ not all equal.

- If $\emptyset = \alpha_0$ has label $\langle a, b, \infty\rangle$ with $a, b \in \omega$, let r_0 be any extension of q having $\sigma^{r_0}(a, b)$, $(l^{r_0}(a))_0$, and $(l^{r_0}(b))_0$ not all equal.
- Otherwise, let $q_0 = q$.

We then proceed by induction. Suppose we have defined r_n and α_n and that the latter is not terminal in T_1^L. Recall that r_n forces that there is some $m \in \omega$ such that $\Phi^f(x, y) = 1$ for $x \in \mathsf{ran}(\alpha_n)$ and $y \geq m$. Let S be the set of all successors $\alpha_n * x$ of α_n with $x \geq m$ and $\mathsf{button}(x) > |r_n|$. The induction breaks into cases according as α_n is or is not a transition node. At the first suitable transition node, we set up to diagonalize.

Case II.1 (Non-Transition Node)

If α_n is not a transition node, let $P = \{x : \alpha_n * x \in S\}$. We choose any $x^* \in P$ and any extension r of r_n which presses $\mathsf{button}(x^*)$, and let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = x^*$.

Case II.2 (Transition Node)

If α_n is a transition node, the induction breaks into four subcases. As previously, the header for each case gives a shorthand for the sort of transition being discussed. The real work of the construction, making full use of the triple labeling of T_1^L, takes place in subcases 3 and 4. The reason for using 3 labels is that later on in the construction we will need to use the pigeonhole principle on numbers we obtain from transition nodes and using 3 labels provides us with more transition nodes along a single branch of the labeled subtree T_1^L.

Case II.2.1 ($\langle\infty, \infty, \infty\rangle \to \langle a, b, \infty\rangle$ or $\langle a, b, c\rangle$)

If α_n has label $\langle\infty, \infty, \infty\rangle$ and every successor of α_n has a label in which the symbol ∞ appears at most once, then we let

$$P = \{x : \alpha_n * x \in S \wedge \forall j < n \ (a_x, b_x > \mathsf{button}(\alpha_n(j))) \wedge a_x, b_x > |r_n|)\},$$

where a_x, b_x here denote the first two entries in the label of $\alpha * x$. We choose any $x^* \in P$ and any extension r of r_n such that, if the label of x^* is $\langle a^*, b^*, c^*\rangle$ with $a^*, b^* \in \omega$ and $c^* \in \omega \cup \{\infty\}$, then r presses $\mathsf{button}(x^*)$ and $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal. Recall the definition of S, whereby we know that

$b_x > |r_n|$ for each $x \in P$. Thus for all $x \in P$, l^{r_n} does not commit us to any particular limiting color for b_x. This is why we are free to press the button of x^*. Let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = x^*$ with label $\langle a^*, b^*, c^* \rangle$. As α_{n+1} is not a transition node, we now return to Case II.1.

Case II.2.2 ($\langle a, \infty, \infty \rangle \to \langle a, b, c \rangle$)

If α_n has label $\langle a^*, \infty, \infty \rangle$ and every successor of α_n has a label in which only finite numbers appear, then we let

$$P = \{x : \alpha_n * x \in S \wedge \forall j < n \ (b_x, c_x > \mathsf{button}(\alpha_n(j))) \wedge b_x, c_x > |r_n|)\},$$

where b_x, c_x here denote the second and third entries in the label of $\alpha * x$. We choose any $x^* \in P$ and any extension r of r_n that presses $\mathsf{button}(x^*)$ and is such that, if the label of x^* is $\langle a^*, b^*, c^* \rangle$ with $a^*, b^*, c^* \in \omega$, then $\sigma^r(b^*, c^*)$, $(l^r(b^*))_0$, and $(l^r(c^*))_0$ are not all equal. Let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = x^*$ with label $\langle a^*, b^*, c^* \rangle$. As α_{n+1} is not a transition node, we now return to Case II.1.

For the following two subcases, an additional consideration is necessary. When we constructed the labeled subtree T_1^L, we fixed the choice of the triple $\langle a, b, c \rangle$ by which a terminal node α is labeled to be the lexically least such triple such that $c > b > a \geq |q|$ and

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(c) \downarrow = 1$$

for some $F_L, F_R \subseteq \mathsf{ran}(\alpha)$. However, we did not fix the choice of F_L and F_R, nor did we record any information about such a choice. Now, however, we need to observe some information about the choice of F_L and F_R. Recall that F_L and F_R represent extensions to the finite approximations of the left and right columns, respectively, of a p-homogeneous set. Per the definition of a p-homogeneous set, if two numbers $x, y \in \omega$ are in the same column of a p-homogeneous set, then the color of the set $\{x, y\}$ can be either 0 or 1 without consequence. This is in contrast to the definition of a homogeneous set, and it is this difference that will give us the freedom to diagonlize against a (strong computable) reduction of SRT^2_2 to SPT^2_2. To that end, we make the following definition, which is intended to formalize the idea that for purposes of the p-homogeneity of a set we are building with respect to a coloring we are building, we can arrange that the color assigned to $\{x, y\}$ does not matter.

Definition 3.6. Let α be a node in the labeled subtree and let $x, y \in \mathsf{ran}(\alpha)$. If α is terminal, we say that x and y *can be placed in the same column* and that $\Phi^f(x, y)$ is *free* if α has label $\langle a, b, c\rangle$ and there are sets $F_L, F_R \subseteq \mathsf{ran}(\alpha)$ satisfying the labeling definition such that one of the following is true:

- $x \in F_L$ and $y \in F_L$;
- $x \in F_R$ and $y \in F_R$;
- $x \notin F_L \cup F_R$ or $y \notin F_L \cup F_R$.

If α is not terminal, we say that x and y can be placed in the same column and that $\Phi^f(x, y)$ is free if there are infinitely many terminal nodes extending α and satisfying the above conditions.

Case II.2.3 ($\langle a, \infty, \infty\rangle \to \langle a, b, \infty\rangle$)

If α_n has label $\langle a^*, \infty, \infty\rangle$ and successor of α_n has a label in which the symbol ∞ appears exactly once, then we let

$$P = \{y : \alpha_n * y \in S \wedge \forall j < n\,(b_y > \mathsf{button}(\alpha_n(j))) \wedge b_y > |r_n|)\},$$

where b_y here denotes the second entry in the label of $\alpha_n * y$. Whether there is much work to be done in this case depends on whether the first node to have label $\langle a^*, \infty, \infty\rangle$ was or was not the root node of T_1^L. Formally, suppose k is the least index such that α_k has a label in which the symbol ∞ appears exactly twice. Then there are two cases.

Case II.2.3.1 ($\alpha_k = \alpha_0 = \emptyset$)

If, on the one hand, $\alpha_k = \alpha_0 = \emptyset$, then we choose any $y^* \in P$ and any extension r of r_n which presses $\mathsf{button}(y^*)$ and is such that, if the label of y^* is $\langle a^*, b^*, \infty\rangle$, then $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal. In this case let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = x^*$ with label $\langle a^*, b^*, \infty\rangle$.

Case II.2.3.2 ($k > 0$ and $\alpha_k \neq \emptyset$)

If, on the other hand, $k > 0$ and $\alpha_k \neq \emptyset$, then there is more work to do. Suppose that $\alpha_k(k-1) = x^*$. Let $P' \subseteq P$ contain precisely the elements y of P which can be placed in the same column as x^*. This is where we begin to use the triple labels of T_1^L. Either $P' \neq \emptyset$ and we diagonalize in this case, or else $P' = \emptyset$ and we wait until the next case to diagonalize, but we are guaranteed to succeed when we attempt in the next case to find a pair of elements which can be placed in the same column.

The important idea here is intuitively as follows. Either we may choose y^* from P that can be placed in the same column of the p-homogeneous set as x^*, or else x^* and y^* cannot be placed in the same column and then a z^* may be chosen at the next transition node that can be placed in the same column as one of x^* or y^*. In either case, the color that Φ^f assigns to the pair of numbers which can be placed in the same column can safely be changed without disturbing the construction.

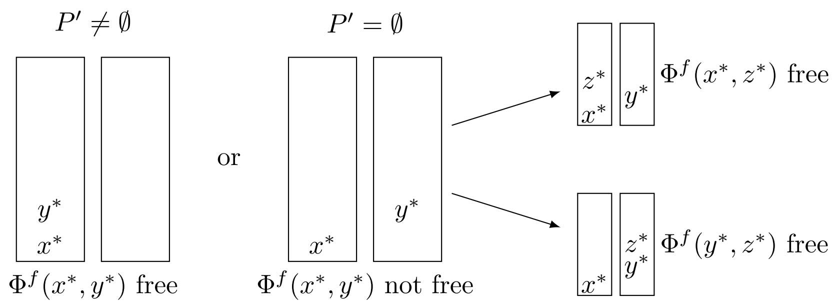

Figure 2. Pairs of adjacent rectangles represent columns of the (finite approximation of the) p-homogeneous set under construction. If $P' \neq \emptyset$, then $\Phi^f(x^*, y^*)$ is free. Otherwise it must be the case that either $\Phi^f(x^*, z^*)$ is free or that $\Phi^f(y^*, z^*)$ is free.

If P' is nonempty, then we choose any $y^* \in P'$ with label $\langle a^*, b^*, \infty\rangle$ and a condition r which extends r_n except possibly having $(l^r(a^*))_1 \neq (l^{q_n}(a^*))_1$ if the latter is defined; and we choose y^* and r such that r presses $\mathsf{button}(y^*)$ and $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal; and we let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = y^*$ with label $\langle a^*, b^*, \infty\rangle$. Finally, if γ is any extension of α_{n+1} in T_1^L such that x^* and y^* cannot be placed in the same column, then we delete from T_1^L γ and all of its extensions. As α_{n+1} is not a transition node, we now return to Case II.1.

Such y^* and r exist in this case for the following reason. Observe that for any $y^* \in P'$ with label $\langle a^*, b^*, \infty\rangle$, $a^* \neq \mathsf{button}(\alpha_n(j))$ for $j \neq k-1$. Since $b^* > |r_n|$, $\sigma^{r_n}(a^*, b^*)$ is not yet defined, so we may choose r extending r_n with $\sigma^r(a^*, b^*) \neq (l^r(a^*))_0$—unless $l^{r_n}(a^*)$ is defined and $(l^{r_n}(a^*))_1 \leq b^*$. In this latter case we let $(l^r(a^*))_0 = (l^{r_n}(a^*))_0$ but choose $(l^r(a^*))_1 > b^*$. Note that in this case r is not an extension of r_n but that r does extend p_s. Hereafter we extend r rather than r_n.

To conclude the argument in this case, it remains to observe that changing the stabilization point of a^* as above does not injure our construction in any way. For the colors of pairs of elements of α_n are forced by facts about σ^{q_n} alone. Furthermore, the fact that $\Phi^f(u, y^*) = 1$ for all $u \in \mathsf{ran}(\alpha_n)$, $u \neq x^*$ with parity unequal to that of y^* is settled by facts about the stabilization points of such numbers u, and these are unaffected by altering $(l^{r_n}(a^*))_1$ since a^* is not the button of any such u. This means that choosing $(l^r(a^*))_1 \neq (l^{r_n}(a^*))_1$ can at most interfere with the stabilization point of x^* and keep us from forcing $\Phi^f(x^*, y^*) = 1$. But note that this does not matter for purposes of the p-homogeneous set we are building, since x^* and y^* will not appear in different columns of the p-homogeneous set we are building and so their mutual color is irrelevant.

If on the other hand P' is empty, then we proceed as in the non-transition case and will diagonalize in Case II.2.4 instead.

Case II.2.4 ($\langle a, b, \infty\rangle \to \langle a, b, c\rangle$)

If α_n has label $\langle a^*, b^*, \infty\rangle$ and every successor of α_n has a label in which only finite numbers occur, and if we failed to diagonalize at an earlier node in the previous case, then we proceed as follows; otherwise we proceed as in the non-transition case. Let

$$P = \{z : \alpha_n * z \in S \wedge \forall j < n\, (c_z > \mathsf{button}(\alpha_n(j))) \wedge c_z > |r_n|)\},$$

where c_z here denotes the third entry in the label of $\alpha_n * z$. Suppose k is the least index such that α_k has a label in which the symbol ∞ appears exactly twice and that $l > k$ is the least index such that α_l has a label in which the symbol ∞ appears exactly once. Such k and l exist by our assumption that we tried and failed to diagonalize in Case II.2.3.2 above. As described in that case, we now capitalize on the fact that we have already seen two transition nodes along the branch of T_1^L that we are building and thus are in a position to apply the pigeonhole principle. We proceed as follows.

Suppose that $\alpha_k(k-1) = x^*$ and that $\alpha_l(l-1) = y^*$. Let $P' \subseteq P$ contain precisely those elements of P which share a column with x^* and $P'' \subseteq P$ contain precisely those elements of P which can be placed in the same column as y^*. At least one of P', P'' must be nonempty; without loss of generality we assume that P'' is nonempty. Then we choose $z^* \in P''$ with label $\langle a^*, b^*, c^*\rangle$ and a condition r which extends r_n except possibly having $(l^r(b^*))_1 \neq (l^{q_n}(b^*))_1$ if the latter is defined; and we choose z^* and r such that $\sigma^r(b^*, c^*)$, $(l^r(b^*))_0$, and $(l^r(c^*))_0$ are not all equal and r presses $\mathsf{button}(z^*)$. We let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = z^*$ with label $\langle a^*, b^*, c^*\rangle$. Such z^* and r exist by the same reasoning given in the previous case. Finally, if γ is any extension of α_{n+1} in T_1^L such that y^* and z^* cannot be placed in the same column, then we delete from T_1^L γ and all of its extensions. As α_{n+1} is not a transition node, we now return to Case II.1.

Completing the Construction

We complete stage s as follows. If added some set P to Y, or if we defined I_{s+1} to be the range of an infinite path through T_0 or T_1, we are done. For if we added some set P to Y, then there will be an $(f \oplus P)$-computable set p-homogeneous for Φ^f, but by Lemma 2.10 there will be no $(f \oplus P)$-computable set homogeneous for f. And if we defined I_{s+1} to be the range of an infinite path through T_0 or T_1—say through T_0—then from the definition of that tree $\Gamma^{H_0^\Phi}$ does does not define an infinite set, let alone one homogeneous for f.

Otherwise, we succeeded either in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_0^L or else in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_1^L; say we succeeded in defining the sequence of nodes in T_0^L. That tree was in this case well-founded, so for some n, α_n was terminal. Then from the definition of the tree, there are some $F_L, F_R \subseteq \mathsf{ran}(\alpha_n)$ such that $\Gamma^{H_{0,s}^\Phi \cup F_L \oplus F_R}(a) \downarrow = \Gamma^{H_{0,s}^\Phi \cup F_L \oplus F_R}(b) \downarrow = 1$ for some unequal $a, b \geq |p_s|$, say with use u. Let $p_{s+1} = q_n$, $H_{0,s+1}^\Phi = H_{0,s}^\Phi \cup F_L \oplus F_R$, and $I_{s+1} = \{x \in I_s : x > u\}$. $\square$

4 Separating SPT^2_2 and SIPT^2_2

In this section we will prove the following theorem, which has the Second Main Theorem as its corollary.

Theorem 4.1. There exists a stable 2-coloring of pairs $f : [\omega]^2 \to 2$ and a family Y of infinite sets such that no $(f \oplus P)$-computable set is p-homogeneous for f for any $P \in Y$, and every stable 2-coloring of pairs $f' : [\omega]^2 \to 2$ computable from f has either an $(f \oplus P)$-computable increasing p-homogeneous set for some $P \in Y$, or if not then some increasing p-homogeneous set which does not compute a set p-homogeneous for f.

Before proving Theorem 4.1, we observe how the Second Main Theorem is a direct consequence. The explanation is the same as that given in the for the deduction of the First Main Theorem from Theorem 3.1.

Proof of Theorem 3.1. We build

- a sequence of $\mathbb{C}$-conditions $p_0 \geq p_1 \geq p_2 \geq \cdots$ with $\lim_s |p_s| = \infty$;
- sequences of finite sets (initial segments of increasing p-homogeneous sets) $H^\Phi_{j,0} \subseteq H^\Phi_{j,1} \subseteq H^\Phi_{j,2} \subseteq \cdots$ for each Turing functional Φ and each $j < 2$;
- a sequence of infinite sets (reservoirs) $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $H^\Phi_{j,s} < I_s$ for each Φ, j, s;
- a sequence of finite families $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$ of infinite subsets of ω;

and we define $f = \bigcup_s \sigma^{p_s}$, $H^\Phi_j = \bigcup_s H^\Phi_{j,s}$ for each $j < 2$, and $Y = \bigcup_s Y_s$. The construction will ensure the following requirements, for all $i \in \omega$ and Turing functionals Φ, Γ, Δ:

$\mathcal{P}_i$: the sequence $p_0 \geq p_1 \geq p_2 \geq \cdots$ is 3-generic relative to each $P \in Y_i$;

$\mathcal{Q}_{\Phi,i}$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$-computable increasing p-homogeneous set for some $P \in Y$ or else both H^Φ_0 and H^Φ_1 are infinite in both columns;

$\mathcal{R}^\Phi_{\Gamma,\Delta}$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$-computable increasing p-homogeneous set for some $P \in Y$; or else if $\Gamma^{H^\Phi_0}$ defines an infinite set then this set is not p-homogeneous for f; or else if $\Delta^{H^\Phi_1}$ defines an infinite set then this set is not p-homogeneous for f.

By way of explaining the $\mathcal{Q}$ and $\mathcal{R}$ requirements, note that by Lemma 2.10, if there is any sequence satisfying the $\mathcal{P}$ requirements and such that for some $P \in Y$ $f \oplus P$ computes a set increasing p-homogeneous for Φ^f, there will then be a set which is increasing p-homogeneous for Φ^f but which computes no set p-homogeneous for f.

4.1 Outline of the Construction

Most of the remarks we made in subsection 3.1 apply *mutatis mutandis*, and we refer the reader to that subsection for additional details. In brief, we will construct an instance f of SPT^2_2 and—via tree labeling—for every instance Φ^f of SIPT^2_2 computed from f a solution H to Φ^f which does not compute any solution to f.

As before, the work of finding the correct extensions for successive finite approximations to the increasing p-homogeneous sets (which appears in the proof as the work to satisfy the $\mathcal{R}$ requirements) will be accomplished using tree labeling constructions. We will attempt to find such extensions for the increasing p-homogeneous set of color 0 by searching through a labeled subtree with 2 labels, and if this fails we will succeed in finding such an extension for the increasing p-homogeneous set of color 1 by searching through a labeled subtree with 3 labels. It is here that we leverage the combinatorial differences between p-homogeneous and increasing p-homogeneous sets. As in the previous proof, we will take leverage these differences by a pigeonhole argument.

Once again, cases beget subcases and subsubcases. There will be a Case I for when the first tree labeling construction succeeds and a Case II for when the first tree labeling construction fails and another is required. Then, under Case I or Case II, there are different concerns depending on whether we are, at a given step of the search through the labeled subtree, looking at a transition node or a non-transition node, as defined in Definition 2.7. The work to be done at non-transition nodes is straightforward, while at transition nodes the work divides further into cases depending on precisely which kind of transition node occurs. In one of these transition node cases, there will be an additional division into subcases based on some technical details of the tree labeling scheme.

While the proof in this section will be more involved than the proof of the First Main Theorem, some parts of the construction remain unchanged. For cases where the construction is identical, we will merely cite the proof of the First Main Theorem. Throughout, we will abide once more by the notational conventions laid out in Remark 3.2.

4.2 Construction

Devote infinitely many stages $s \in \omega$ to each requirement. Let p_0 be any condition with $|p_s| = 0$. For each Φ let $H^\Phi_{0,0} = H^\Phi_{1,0} = \emptyset$, and let $I_0 = \omega$ and $Y_0 = \emptyset$. At stage $s+1$ assume by way of induction that we have p_s, $H^\Phi_{j,s}$ for $j < 2$ and all Φ, I_s, and Y_s and assume that if $H^\Phi_{j,s}$ is nonempty for some j and Φ, then p_s forces that Φ^f is a stable coloring of pairs and that for $x < y$, $\Phi^f(x, y) = j$ whenever $2x, 2y + 1 \in H^\Phi_{j,s}$ or when $y \in I_s$ and $2x \in H^\Phi_{j,s}$. At the end of a stage any of p_{s+1}, $H^\Phi_{j,s+1}$, I_{s+1}, or Y_{s+1} not yet defined should be taken to be identical to p_s, $H^\Phi_{j,s}$, I_s, or Y_s, respectively.

$\mathcal{P}$ requirements

Suppose s is dedicated to requirement $\mathcal{P}_i$ for some $i < s$ and that it is the $\langle n, m\rangle^{\text{th}}$ such stage. If $n > |Y_i|$ do nothing. Otherwise, let P be the n^{th} member of the family Y_i in some fixed enumeration and let W be the m^{th} $\Sigma^0_3(P)$ set in some fixed enumeration. If p_s has an extension q in W, fix q and let $p_{s+1} = q$, so that $p_0 \geq p_1 \geq \cdots \geq p_s \geq p_{s+1} \geq \cdots$ meets W. Otherwise, do nothing, and $p_0 \geq p_1 \geq \cdots \geq p_s \geq p_{s+1} \geq \cdots$ avoids W.

$\mathcal{Q}$ requirements

Suppose s is dedicated to $\mathcal{Q}_{\Phi,i}$. Without loss of generality we assume that p_s decides whether or not Φ^f is a stable 2-coloring of pairs. If p_s forces that Φ^f is not such a coloring, do nothing. Otherwise, we consider two cases.

- If for some $j < 2$ and $k \in \omega$ there is no extension of p_s which forces that $\lim_u \Phi^f(x, u) = j$ for some $x \geq k$ in I_s, then $P = \{x \in I_s : x \geq k\}$ is limit homogeneous for Φ^f with color $1 - j$, and so $(f \oplus P)$ computes an increasing p-homogeneous set for Φ^f. We set $Y_{s+1} = Y_s \cup \{P\}$. This satisfies the requirement.

- If no such j, k exist, then there are numbers $x_{00}, x_{01}, x_{10}, x_{11} \in I_s$ with $x_{00} < x_{01}$ and $x_{10} < x_{11}$ and an extension of p_s forcing that $H^\Phi_{j,s} \cup \{2x_{j0}, 2x_{j1} + 1\}$ is finite increasing p-homogeneous for Φ^f and $\lim_u \Phi^f(x_{ji}, u) = j$ for each $i, j < 2$. In this case let p_{s+1} be such an extension of p_s, let $H^\Phi_{j,s+1} = H^\Phi_{j,s} \cup \{2x_{j0}, 2x_{j1} + 1\}$, and let $I_{s+1} = \{x : m < x \in I_s\}$ where m is greater than the stabilization points under Φ^f of every element of $H^\Phi_{0,s+1} \cup H^\Phi_{1,s+1}$. Observe that both columns have been extended by one element.

$\mathcal{R}$ requirements

Suppose s is dedicated to $\mathcal{R}^\Phi_{\Gamma,\Delta}$ and assume that p_s forces that Φ^f is a stable coloring of pairs. The goal of this requirement is to extend by some finite set either the initial segment of H^Φ_0 (Case I) or the initial segment of H^Φ_1 (Case II) that we have constructed so far, subject to the following condition: if we extend H^Φ_0, then Γ does not compute a p-homogeneous set for f from any further extension of H^Φ_0; and if on the other hand we extend H^Φ_1, then Δ does not compute a p-homogeneous set for f from any further extension of H^Φ_1. When we have so extended one of H^Φ_0, H^Φ_1, we will say that we have successfully *diagonalized* against such computations.

We intend to accomplish this diagonalization in the following way. There will be two numbers, say a and b, and two finite sets F_L and F_R, all arising from a tree labeling construction, about which we know either that

$$\Gamma^{H^\Phi_0 \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H^\Phi_0 \cup (F_L \oplus F_R)}(b) \downarrow = 1$$

or else that

$$\Delta^{H^\Phi_1 \cup (F_L \oplus F_R)}(a) \downarrow = \Delta^{H^\Phi_1 \cup (F_L \oplus F_R)}(b) \downarrow = 1.$$

Let us suppose we know the first. In other words, we know that if we extend H^Φ_0 by $F_L \oplus F_R$, then the set computed from $H^\Phi_0 \cup (F_L \oplus F_R)$ by Γ will contain a and b. Thus to diagonalize—i.e. to ensure that the set so computed is not homogeneous for f—we will choose an extension q of the condition p_s such that $\sigma^q(a,b)$, $(l^q(a))_0$, and $(l^q(b))_0$ are not all equal. This will guarantee that no set containing both a and b is homogeneous for the coloring eventually obtained by extending σ^q. The tension of the proof arises from the fact that, while diagonalizing in this way, we also need q to force that elements of F_L and F_R have the right limits under Φ^f so that $H^\Phi_0 \cup (F_L \oplus F_R)$ can in fact be extended to a p-homogeneous set.

The means by which we accomplish such a diagonalization here will be slightly more involved than in the previous section. This is because we are trying to leverage the combinatorial differences between a p-homogeneous set and an increasing p-homogeneous set rather than between a p-homogeneous set and a homogeneous one. Because both of the kinds of homogeneous set under consideration have two columns and the distinction has to do with the order of the elements in those columns, it will be important to keep track of the way we place numbers into each column during the construction. So in the present proof there will again be two numbers, say a and b, and two finite sets F_L and F_R, all arising from a tree labeling construction, about which we know either that

$$\Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(2a+1) \downarrow = \Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(2b) \downarrow = 1$$

or else that

$$\Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(2a+1) \downarrow = \Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(2b) \downarrow = 1.$$

Let us suppose we know the first. In other words, we know that if we extend H_0^Φ by $F_L \oplus F_R$, then the set computed from $H_0^\Phi \cup (F_L \oplus F_R)$ by Γ will contain $2a+1$ and $2b$. When we view the set so computed as having two columns (i.e. as being the join of two sets), this means that a appears in the right-hand column and b appears in the left-hand column. Thus to ensure that the set so computed is not p-homogeneous for f, we will choose as before an extension q of the condition p_s such that $\sigma^q(a,b)$, $(l^q(a))_0$, and $(l^q(b))_0$ are not all equal. This will guarantee that no set containing both a and b is p-homogeneous for the coloring eventually obtained by extending σ^q.

Case I (Extending H_0^Φ)

For this proof we need again to modify slightly the definition of tree labeling with two labels. We now say that a nonempty string $\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \mathsf{ran}(\alpha^\#)$ and no $b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(2a+1) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(2b) \downarrow = 1.$$

With this modification of the tree labeling definition in hand, we now as before let $T_0 = T(|p_s|, \Gamma, H_{0,s}^\Phi, I_s)$. If T_0 is not well-founded then let I_{s+1} be the range of an infinite path through T_0. Observe that in this case the requirement is satisfied. If T_0 is well-founded, then let T_0^L be the labeled subtree of T_0.

We now try to define two sequences, conditions

$$p_s \geq q_0 \geq q_1 \geq q_2 \geq \cdots$$

and nodes of T_0^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \cdots$$

where for all $j \geq 0$, α_{j+1} is a successor of α_j and for all $j \geq 0$ the condition q_j forces that

$$\lim_u \Phi^f(x, u) = 0$$

for all $x \in \mathsf{ran}(\alpha_j)$. The remainder of the argument for this case is exactly the same as that for Case I in Theorem 3.1.

Case II (Extending H_1^Φ)

We proceed as in the previous section. Recall that this is the case we are in after having already attempted Case I, but having failed in Case I.2.2. Thus we begin the present case with information that will allow us to perform a tree labeling construction with 3 labels with no possibility of failure. Recall that we have assumed that p_s (and hence q_n) forces that Φ^f is a stable 2-coloring of pairs,and recall that Case I.2.2 failed because we could not find a tuple $\langle x^*, a^*, b^* \rangle \in P$ and an extension q of q_n which forces that $\lim_u \Phi^f(x^*, u) = 0$ and is such that $\sigma^q(a^*, b^*)$, $(l^q(a^*))_0$, and $(l^q(b^*))_0$ are not all equal. This means, given the definitions above of Q and the revised reservoir I_s, that for any $x \in I_s$ and any extension q of q_n such that $\sigma^q(a^*, b^*)$, $(l^q(a^*))_0$, and $(l^q(b^*))_0$ are not all equal q forces that $\lim_u \Phi^f(x^*, u) = 1$. As before, we will make use of Definition 3.3 and Definition 3.4 and make the notational adjustment of Remark 3.5.

Here again we must modify the tree labeling method, varying tree labeling with three labels. The definition of $T(k, \Gamma, H, I)$ is for this case changed to the following. $\emptyset \in T(k, \Gamma, H, I)$ and for a nonempty string α, $\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \mathsf{ran}(\alpha^\#)$ and no $c > b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(2a+1) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(2b) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(2c+1) \downarrow = 1.$$

The method for labeling the nodes of $T(k, \Gamma, H, I)$ extends the method from tree labeling with two labels in the natural way, as does the method for selecting the nodes of the labeled subtree $T^L(k, \Gamma, H, I)$. Remark 2.3 applies *mutatis mutandis*.

Now we begin a new tree labeling construction starting where we left off when Case I.2.2 failed. Thus we have the forcing condition (formerly designated q_n; recall Remark 3.5) $q \leq p_s$ and the revised reservoir I_s.

Let $T_1 = T(|q|, \Delta, H^{\Phi}_{1,s}, I_s)$. If T_1 is not well founded then let I_{s+1} be the range of an infinite path through T_1. Observe that in this case the requirement is satisfied. If T_1 is well founded, let T_1^L be the labeled subtree of T_1.

We now try to define two sequences, conditions

$$q \geq r_0 \geq r_1 \geq r_2 \geq \cdots$$

and nodes of T_1^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \cdots$$

where for all $j \geq 0$, α_{j+1} is a successor of α_j and for all $j \geq 0$ the condition r_j forces that

$$\lim_u \Phi^f(x, u) = 1$$

for all $x \in \mathsf{ran}(\alpha_j)$. We begin the definition of these sequences as follows.

- If $\emptyset = \alpha_0$ has label $\langle a, b, c\rangle$ with $a, b, c \in \omega$, let r_0 be any extension of q having $\sigma^{r_0}(a, b)$, $(l^{r_0}(a))_0$, and $(l^{r_0}(b))_0$ not all equal.
- If $\emptyset = \alpha_0$ has label $\langle a, b, \infty\rangle$ with $a, b \in \omega$, let r_0 be any extension of q having $\sigma^{r_0}(a, b)$, $(l^{r_0}(a))_0$, and $(l^{r_0}(b))_0$ not all equal.
- Otherwise, let $q_0 = q$.

We then proceed by induction. Suppose we have defined r_n and α_n and that the latter is not terminal in T_1^L. Recall that r_n forces that there is some $m \in \omega$ such that $\Phi^f(x, y) = 1$ for $x \in \mathsf{ran}(\alpha_n)$ and $y \geq m$. Let S be the set of all successors $\alpha_n * x$ of α_n with $x \geq m$ and $\mathsf{button}(x) > |r_n|$. The induction breaks into cases according as α_n is or is not a transition node. At the first suitable transition node, we set up to diagonalize. Hereafter the construction is exactly the same as that of Theorem 3.1 up until Case II.2.3, where the diagonalization strategy changes to reflect the different combinatorics of the present theorem. We therefore refer the reader to the previous section and restart this one at Case II.2.3 below. Before we do so, it is important that we revise Definition 3.6 to reflect the fact that we are building an *increasing* p-homogeneous set.

Definition 4.2 (Definition 3.6 Revised for Increasing p-Homogeneous Sets)**.** Let α be a node in the labeled subtree and let $x, y \in \mathsf{ran}(\alpha)$ with $x < y$. If α is terminal, we say that $\Phi^f(x, y)$ is *free* if α has label $\langle a, b, c\rangle$ and there are sets $F_L, F_R \subseteq \mathsf{ran}(\alpha)$ satisfying the labeling definition such that one of the following is true:

- $x \in F_L$ and $y \in F_L$;
- $x \in F_R$;
- $x \notin F_L \cup F_R$ or $y \notin F_L \cup F_R$.

If α is not terminal, we say that $\Phi^f(x, y)$ is free if there are infinitely many terminal nodes extending α and satisfying the above conditions.

As before, this definition is intended to formalize the idea that for purposes of the increasing p-homogeneity of a set we are building with respect to a coloring we are building, we can arrange that the color assigned to $\{x, y\}$ does not matter.

Case II.2.3 ($\langle a, \infty, \infty\rangle \to \langle a, b, \infty\rangle$)

If α_n has label $\langle a^*, \infty, \infty\rangle$ and successor of α_n has a label in which the symbol ∞ appears exactly once, then we let

$$P = \{y : \alpha_n * y \in S \wedge \forall j < n\, (b_y > \mathsf{button}(\alpha_n(j))) \wedge b_y > |r_n|)\},$$

where b_y here denotes the second entry in the label of $\alpha_n * y$. Whether there is much work to be done in this case depends on whether the first node to have label $\langle a^*, \infty, \infty\rangle$ was or was not the root node of T_1^L. Formally, suppose k is the least index such that α_k has a label in which the symbol ∞ appears exactly twice. Then there are two cases.

Case II.2.3.1 ($\alpha_k = \alpha_0 = \emptyset$)

If, on the one hand, $\alpha_k = \alpha_0 = \emptyset$, then we choose any $y^* \in P$ and any extension r of r_n which presses $\mathsf{button}(y^*)$ and is such that, if the label of y^* is $\langle a^*, b^*, \infty\rangle$, then $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal. In this case let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = x^*$ with label $\langle a^*, b^*, \infty\rangle$.

Case II.2.3.2 ($k > 0$ and $\alpha_k \neq \emptyset$)

If, on the other hand, $k > 0$ and $\alpha_k \neq \emptyset$, then there is more work to do. Suppose that $\alpha_k(k-1) = x^*$. Let $P' \subseteq P$ contain precisely the elements y of P such that $\Phi^f(x^*, y)$ is free. This is where we begin to use the triple labels of T_1^L. Either $P' \neq \emptyset$ and we diagonalize in this case, or else $P' = \emptyset$ and we wait until the next case to diagonalize, but we are guaranteed to succeed when we attempt in the next case to find a pair of elements for which Φ^f is free.

The important idea here is intuitively as follows. Either we may choose y^* from P so that $\Phi^f(x^*, y^*)$ is free, or else $x^* < y^*$ must be in the only configuration such that $\Phi^f(x^*, y^*)$ is not free; that is, x^* must inhabit the left column and y^* the right column. But then since x^* and y^* are in different columns, it is guaranteed that at the next transition node we encounter as we build a branch through T_1^L we may choose a z^* such that either $\Phi^f(x^*, z^*)$ is free or $\Phi^f(y^*, z^*)$ is free.

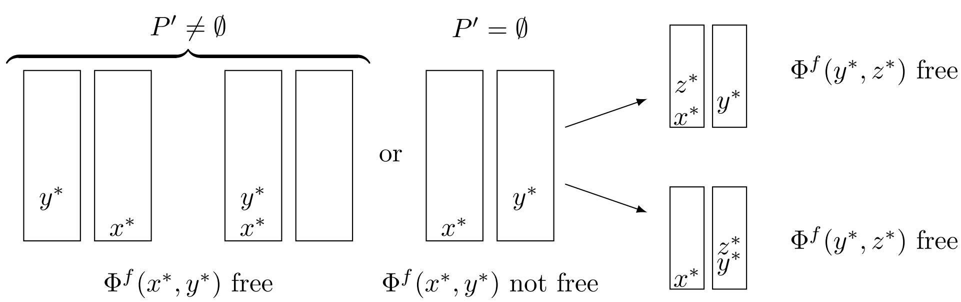

Figure 3. Pairs of adjacent rectangles represent columns of the (finite approximation of the) increasing p-homogeneous set under construction. If $P' \neq \emptyset$, then $\Phi^f(x^*, y^*)$ is free. Otherwise it must be the case that either $\Phi^f(x^*, z^*)$ is free or that $\Phi^f(y^*, z^*)$ is free.

If P' is nonempty, then we choose any $y^* \in P'$ with label $\langle a^*, b^*, \infty\rangle$ and a condition r which extends r_n except possibly having $(l^r(a^*))_1 \neq (l^{r_n}(a^*))_1$ if the latter is defined; and we choose y^* and r such that r presses $\mathsf{button}(y^*)$ and $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal; and we let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$

for any $\beta \in S$ having $\beta(n) = y^*$ with label $\langle a^*, b^*, \infty \rangle$. Finally, if γ is any extension of α_{n+1} in T_1^L such that $\Phi^f(x^*, y^*)$ is not free, then we delete from T_1^L γ and all of its extensions. As α_{n+1} is not a transition node, we now return to Case II.1.

If on the other hand P' is empty, then we proceed as in the non-transition case and will diagonalize in Case II.2.4 instead.

Case II.2.4 ($\langle a, b, \infty \rangle \to \langle a, b, c \rangle$)

If α_n has label $\langle a^*, b^*, \infty \rangle$ and every successor of α_n has a label in which only finite numbers occur, and if we failed to diagonalize at an earlier node in the previous case, then we proceed as follows; otherwise we proceed as in the non-transition case. Let

$$P = \{z : \alpha_n * z \in S \wedge \forall j < n\, (c_z > \mathsf{button}(\alpha_n(j))) \wedge c_z > |r_n|)\},$$

where c_z here denotes the third entry in the label of $\alpha_n * z$. Suppose k is the least index such that α_k has a label in which the symbol ∞ appears exactly twice and that $l > k$ is the least index such that α_l has a label in which the symbol ∞ appears exactly once. Such k and l exist by our assumption that we tried and failed to diagonalize in Case II.2.3.2 above. As described in that case, we now capitalize on the fact that we have already seen two transition nodes along the branch of T_1^L that we are building and thus are in a position to apply the pigeonhole principle. We proceed as follows.

Suppose that $\alpha_k(k-1) = x^*$ and that $\alpha_l(l-1) = y^*$. Let $P' \subseteq P$ contain precisely those elements $z \in P$ such that $\Phi^f(x^*, z)$ is free and $P'' \subseteq P$ contain precisely those elements $z \in P$ such that $\Phi^f(y^*, z)$ is free. At least one of P', P'' must be nonempty; without loss of generality we assume that P'' is nonempty. Then we choose $z^* \in P''$ with label $\langle a^*, b^*, c^* \rangle$ and a condition r which extends r_n except possibly having $(l^r(b^*))_1 \neq (l^{r_n}(b^*))_1$ if the latter is defined; and we choose z^* and r such that $\sigma^r(b^*, c^*)$, $(l^r(b^*))_0$, and $(l^r(c^*))_0$ are not all equal and r presses $\mathsf{button}(z^*)$. We let $r_{n+1} = r$ and $\alpha_{n+1} = \beta$ for any $\beta \in S$ having $\beta(n) = z^*$ with label $\langle a^*, b^*, c^* \rangle$. Such z^* and r exist by the same reasoning given in the proof in the previous section. Finally, if γ is any non-terminal extension of α_{n+1} in T_1^L such that $\Phi^f(y^*, z^*)$ is not free, then we delete from T_1^L γ and all of its extensions. As α_{n+1} is not a transition node, we now return to Case II.1.

Completing the Construction

We complete stage s as follows. If added some set P to Y, or if we defined I_{s+1} to be the range of an infinite path through T_0 or T_1, we are done. Otherwise, we succeeded either in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_0^L or else in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_1^L; say we succeeded in defining the sequence of nodes in T_0^L. This tree was in this case well-founded, so for some n, α_n was terminal. Then from the definition of the tree, there are some $F_L, F_R \subseteq \mathsf{ran}(\alpha_n)$ such that $\Gamma^{H^{\Phi}_{0,s} \cup (F_L \oplus F_R)}(2a+1) \downarrow= \Gamma^{H^{\Phi}_{0,s} \cup (F_L \oplus F_R)}(2b) \downarrow= \Gamma^{H^{\Phi}_{0,s} \cup (F_L \oplus F_R)}(2c+1) \downarrow= 1$ for some unequal $a, b, c \geq |p_s|$, say with use u. Let $p_{s+1} = r_n$, $H^{\Phi}_{0,s+1} = H^{\Phi}_{0,s} \cup F_L \oplus F_R$, and $I_{s+1} = \{x \in I_s : x > u\}$. $\square$

5 Summary of Results

We selected a particular "menagerie" within the reverse mathematics zoo, consisting of the four stable relatives of Ramsey's theorem for 2-colorings of pairs SRT^2_2, SPT^2_2, SIPT^2_2, and D^2_2. We know [3] that these four principles are all logically equivalent over RCA_0, but the finer structure of the menagerie under strong reducibility was largely unknown. It follows from a lemma of Cholak, Jockusch, and Slaman [2] that all four theorems are computably equivalent, and Dzhafarov [9] proved that $\mathsf{SRT}^2_2 \not\leq_{\mathrm{W}} \mathsf{D}^2_2$ and $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{D}^2_2$. We set out to determine which of the other possible reductions succeed or fail. While some of the reductions are trivial, in particular the non-reductions $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SIPT}^2_2$ are nontrivial and to prove these we required several elaborations of the tree labeling constructions used in previous work [9, 13].

In summary, we have shown that while $\mathsf{D}^2_2 \leq_{\mathrm{sW}} \mathsf{SIPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SPT}^2_2 \leq_{\mathrm{sW}} \mathsf{SRT}^2_2$, $\mathsf{SRT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{SIPT}^2_2 \not\leq_{\mathrm{sc}} \mathsf{D}^2_2$. This formalizes our intuitions about the relative complexity of these four principles and shows that $\leq_{\mathrm{sc}}$ is the right level of strong reduction needed to separate these principles. Figure 4 records the status of all the strong reductions between the principles studied in this paper. In the diagram, we write $Q \to P$ to mean that problem P reduces to problem Q in the indicated sense.

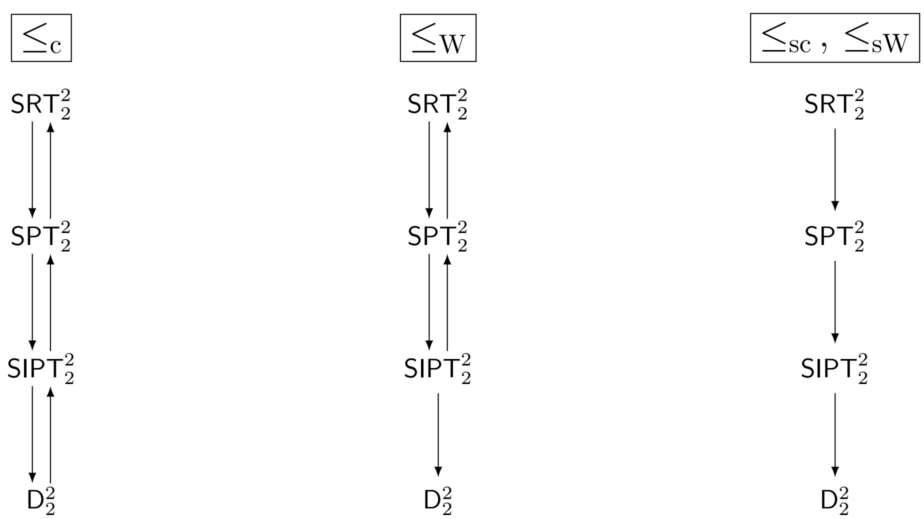

Figure 4. Complete map of the strong reductions which hold between the four principles. An arrow from P to Q indicates that Q is reducible in the given sense to P. Arrows are absent where the corresponding reduction fails.

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