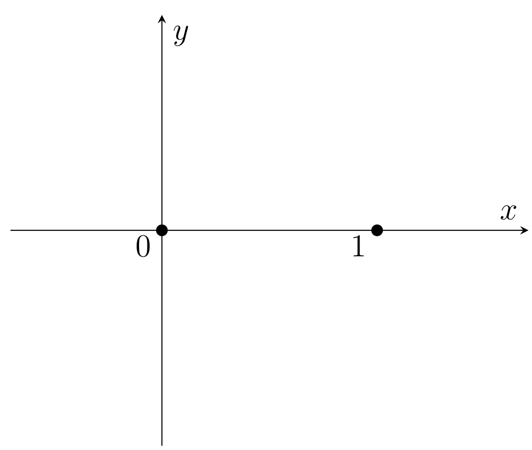
1. Compute $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$.

We evaluate iterated integrals from the inside out. So the first step to computing the above iterated integral is to find $\int_x^1 e^{x/y}\, dy$. That, however, is problematic: we have no good way of finding the antiderivative of $e^{c/y}$ for any constant c. In fact, the antiderivative can't be written in terms of elementary functions (functions calculators know the names for). If we came across this integral in Calc II, the best we could do would be to write $e^{x/y}$ as an infinite series, and that won't get us very far here.

So if we can't find $\int_x^1 e^{x/y}\, dy$, what can we do? We have to try to reverse the order of integration. If we can do that, the inner integral will be $\int e^{x/y}\, dx$, which is easy. In order to swap the order of the integrals, we need to look at the region of integration. When you **have** an iterated integral and you **want** to sketch the region of integration, start reading the integrals from the outside in. Here the outer integral says x goes from $x = 0$ to $x = 1$:

$$\int_0^1 \int_x^1 e^{x/y}\, dy\, dx.$$

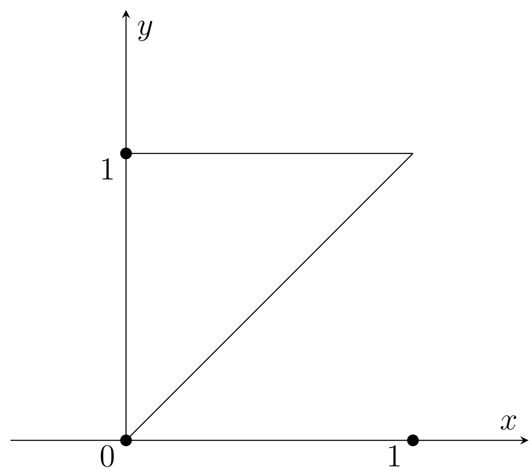
We mark this on our sketch:



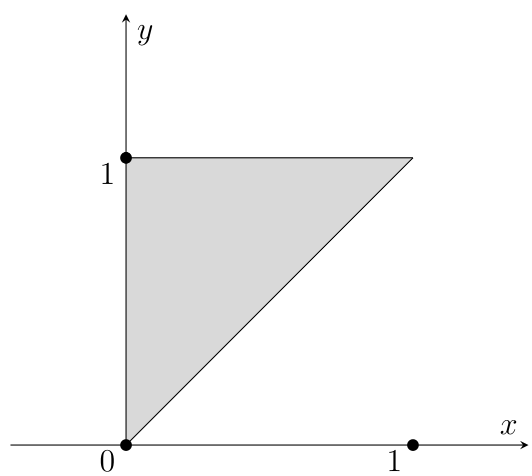
Next we read the inner integral, which says that y ranges from $y = x$ up to $y = 1$:

$$\int_0^1 \int_x^1 e^{x/y}\, dy\, dx.$$

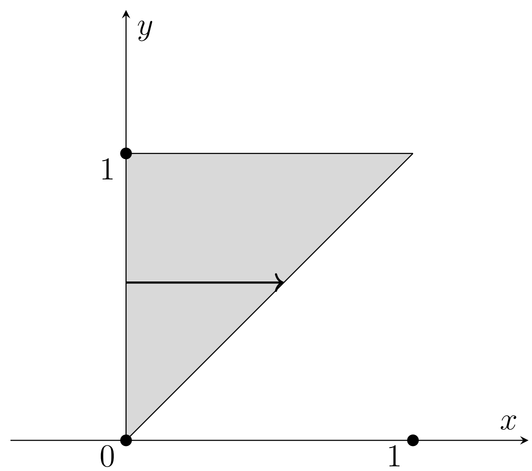
So we draw the lines $y = 1$ and $y = x$ on the sketch:



We're integrating over the region where x is between 0 and 1 and y is between x and 1, i.e. over the shaded region:



Alright, we have our region of integration. Our goal was to be able to use this sketch to reverse the order of integration. So instead of specifying the range of y in terms of x, we specify the range of x in terms of y, as represented by the arrow below in the sketch on the next page:



Remember what we're doing here: we want to write down the original integral with the integrals in the other order, and all we have to do is fill in the limits of integration:

$$\int_{\square}^{\square} \int_{\square}^{\square} e^{x/y}\,dx\,dy.$$

We're thinking of the region as a Type II region now, which is to say that the range of x depends on y and y just ranges between two numbers. Reading our sketch, we see that x ranges from $x = 0$ to $x = y$ and y ranges from $y = 0$ to $y = 1$. That means we can fill in the integral:

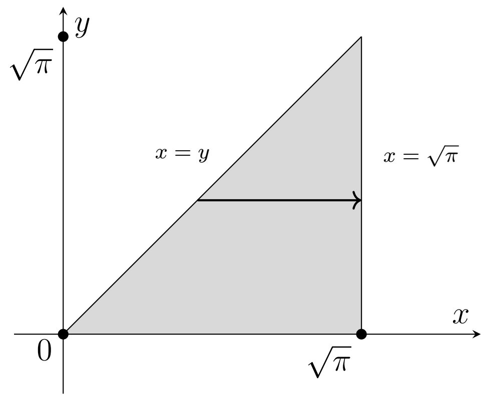
$$\int_0^1 \int_0^y e^{x/y}\,dx\,dy.$$

All that work was just to rewrite the iterated integral in a way that we can evaluate. Now we have to actually find the answer. As always, we evaluate the iterated integral from the inside out:

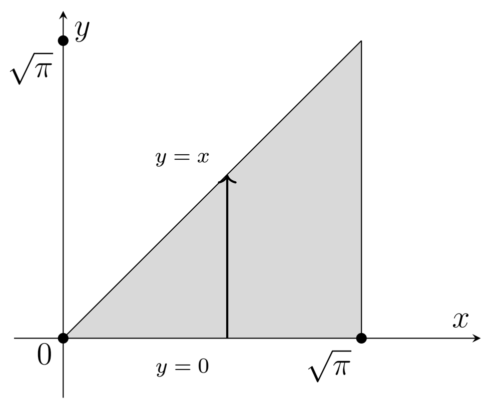
$$\begin{aligned}
\int_0^1 \int_0^y e^{x/y}\,dx\,dy &= \int_0^1 \left[\left(ye^{x/y}\right)\Big|_{x=0}^{x=y} \right] dy \\
&= \int_0^1 \left[ye^{y/y} - ye^{0/y} \right] dy \\
&= \int_0^1 y(e-1)\,dy \\
&= \left(\frac{1}{2}y^2(e-1)\right)\Big|_0^1 = \boxed{\frac{e-1}{2}}.
\end{aligned}$$

2. Evaluate $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy$.

This is another case where the inner integral doesn't work and we have to try to change the order of integration. If we try to evaluate the integral as it stands, we have to find $\int \cos(x^2)\, dx$, which we can't do. However, $\int \cos(x^2)\, dy$ would be really easy to find. So we set about trying to switch the order of integration, and to that end we first draw a sketch. The outer integral tells us that y goes from $y = 0$ to $y = \sqrt{\pi}$, and the inner integral tells us that x goes from $x = y$ to $x = \sqrt{\pi}$. This gives us another triangle:



Now, instead of specifying the range of x in terms of y, we want to reverse things and specify the range of y in terms of x:



We want to write the integral with the order of integration switched:

$$\int_{\square}^{\square} \int_{\square}^{\square} \cos(x^2)\, dy\, dx.$$

Now we're in a position to fill in the limits of integration. Looking at the sketch, we see that x goes from $x = 0$ to $x = \sqrt{\pi}$, giving us

$$\int_0^{\sqrt{\pi}} \int_{\square}^{\square} \cos(x^2)\, dy\, dx.$$

Looking at the sketch again, we see that y goes from $y = 0$ to $y = x$, giving us

$$\int_0^{\sqrt{\pi}} \int_0^{x} \cos(x^2)\, dy\, dx.$$

Now we can evaluate the integral, starting as always with the inside integral:

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_0^{x} \cos(x^2)\, dy\, dx &= \int_0^{\sqrt{\pi}} \left[\left(y \cos(x^2)\right)\Big|_{y=0}^{y=x} \right] dx \\
&= \int_0^{\sqrt{\pi}} \left[x\cos(x^2) - 0\cos(x^2) \right] dx \\
&= \int_0^{\sqrt{\pi}} x \cos(x^2)\, dx \\
&= \frac{1}{2} \int_0^{\pi} \cos(u)\, du \quad (\text{with } u = x^2 \text{ and } du = 2x\, dx) \\
&= \frac{1}{2} \sin(u)\Big|_0^{\pi} = \frac{1}{2}(\sin(\pi) - \sin(0)) = \boxed{0.}
\end{aligned}$$