Change of Variables

Change of variables in multiple integrals is complicated, but it can be broken down into steps as follows. The starting point is a double integral in x & y.

$$\iint_R f(x,y)\,dA$$

(1) If the transformation T hasn't already been given, come up with the transformation to use.

(2) Figure out both directions of the transformation.[1]

(3) Fill in the following information in any order:

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

(a) change R to S

(b) plug in u & v in place of x & y

(c) calculate the Jacobian

Note that depending on the problem you might use either order of integration, $du\,dv$ or $dv\,du$.

Example 1

Use the transformation $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$ to evaluate the integral $\iint_R (4x+8y)\,dA$, where R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$.

1 I have already been given the transformation.

2 To get u & v in terms of x & y, I solve for u and v in the system I was given (notice that I've distributed out the $\frac{1}{4}$ in each equation):

$$x = \frac{1}{4}u + \frac{1}{4}v \qquad (1)$$

$$y = -\frac{3}{4}u + \frac{1}{4}v \qquad (2)$$

Subtracting (2) from (1) gives me

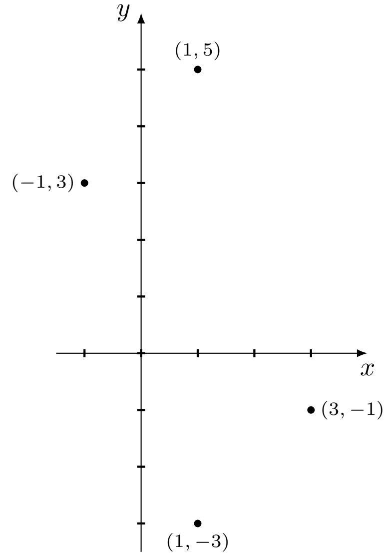
$$\boxed{u = x - y,}$$

and adding three times (1) to (2) gives me

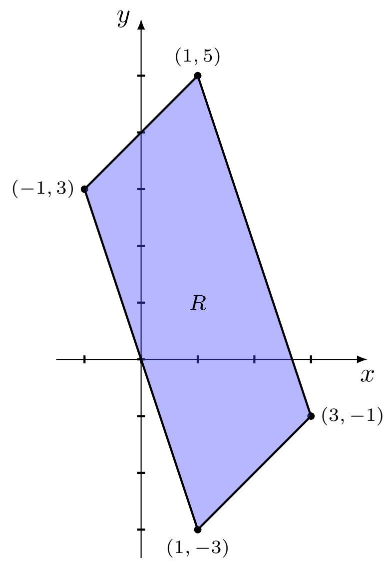
$$\boxed{v = 3x + y.}$$

3(a) This is nearly always the hardest step, and it can require a lot of work. In order to figure out S, I first need to understand what R looks like. So I'm going to draw a picture. It isn't strictly necessary to draw a picture, but I find that if I have a picture to work with, the rest of the problem gets easier. The problem statement has told me that R is a parallelogram with certain corners, so the first step to draw R is to mark the corner points I was given. The result appears on the next page.

[1] We need to know how to get (x,y) from (u,v) to calculate the Jacobian, and we need to know how to get (u,v) from (x,y) to figure out the region of integration.



Then I connect the dots to draw R:



The point of drawing this picture of R (the region of integration on the x, y side of the transformation) was so that I could figure out S (the region of integration over on the u, v side of the transformation). To figure out S, all I have to do is figure out the boundary and then everything else will fill itself in.

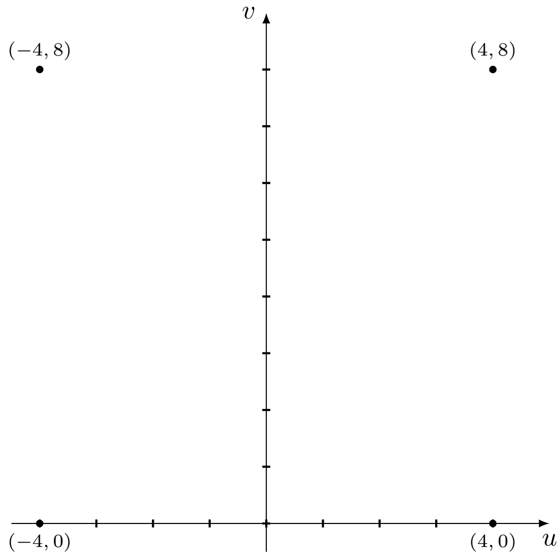
In fact, all I have to is figure out either (i) what the transformation does to each of the 4 sides of R, or else (ii) what the transformation does to each of the 4 corners of R. Once I do either of those, the rest of S will fill itself in. I know from step (2) that

$$\boxed{u = x - y, \quad v = 3x + y.}$$

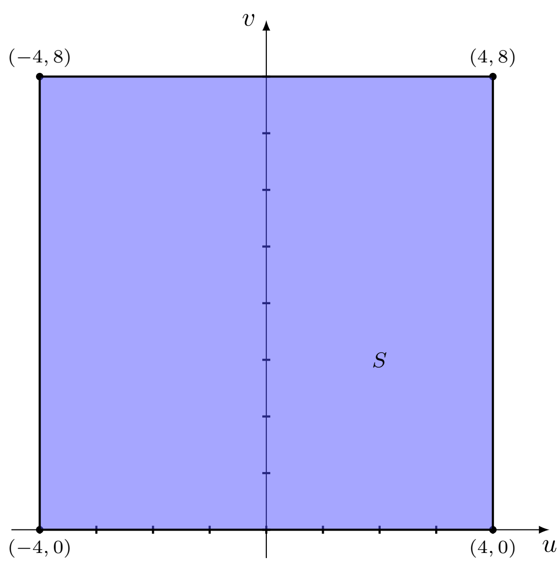
So plugging in the four corner points of R gives me the corners of S:

	(x, y)	(u, v)
Corner 1:	$(-1, 3)$	$(-4, 0)$
Corner 2:	$(1, -3)$	$(4, 0)$
Corner 3:	$(3, -1)$	$(4, 8)$
Corner 4:	$(1, 5)$	$(-4, 8)$

Now I can mark the corners of S...



...and fill in the rest.



I've now got the hardest part out of the way, and I can fill in part of the integral:

$$\iint_R (4x + 8y)\, dA = \int_{-4}^{4} \int_{0}^{8} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du$$

✓ **(a)** change R to S

(b) plug in u & v in place of x & y

(c) calculate the Jacobian

3(b) Since I already know what x & y are as functions of u & v, I can just plug in that information:

$$4x + 8y = 4[x(u,v)] + 8[y(u,v)] = 4\left[\frac{1}{4}(u+v)\right] + 8\left[\frac{1}{4}(v - 3u)\right] = 3v - 5u.$$

Now I can fill in another part of the integral:

$$\iint_R (4x+8y)\,dA = \int_{-4}^{4}\int_0^8 (3u-5v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

✓ **(a)** change R to S

✓ **(b)** plug in u & v in place of x & y

(c) calculate the Jacobian

3(c) The last ingredient I need for the change of variables is the Jacobian. Since I know what x & y are as functions of u & v, I can take the derivatives and plug them in to get the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \mathsf{det}\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \mathsf{det}\begin{bmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{bmatrix} = \frac{1}{16} - \frac{-3}{16} = \frac{1}{4}.$$

This is the last part of the integral that I need to fill in:

$$\iint_R (4x+8y)\,dA = \int_{-4}^{4}\int_0^8 (3u-5v)\left|\frac{1}{4}\right| dv\,du$$

✓ **(a)** change R to S

✓ **(b)** plug in u & v in place of x & y

(c) calculate the Jacobian ✓

Now that I'm done with the change of variables, I can evaluate the integral in the usual way:

$$\iint_R (4x+8y)\,dA = \int_{-4}^{4}\int_0^8 (3u-5v)\left|\frac{1}{4}\right| dv\,du$$

$$= \frac{1}{4}\int_{-4}^{4}\int_0^8 (3u-5v)\,dv\,du \qquad \text{(factor out the 1/4)}$$

$$= \frac{1}{4}\int_{-4}^{4}\left[\frac{3}{2}v^2 - 5uv\right]_{v=0}^{v=8} du \qquad \text{(inside integral first)}$$

$$= \frac{1}{4}\int_{-4}^{4}(96-40u)\,du \qquad \text{(plug in limits)}$$

$$= \frac{1}{4}\Big[96u - 20u^2\Big]_{-4}^{4} \qquad \text{(now the } u \text{ integral)}$$

$$= \boxed{192} \qquad \text{(plug in limits)}$$

Example 2

Make an appropriate change of variables to evaluate the integral $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

1 I need to come up with a change of variables that could make this integral easier. There are two ways I could approach this: I could try to make the integrand nicer or I could try to make the region of integration nicer. Alternatively, I could try to look at both the integrand and the region of integration while I try to come up with the transformation to use.

After staring at the problem for a while, I start to notice some patterns. $(x+y)$ shows up in a lot of places: it's mentioned twice in the description of R, it appears in the integrand, and in fact if I write

$$(x+y)e^{x^2-y^2} = (x+y)e^{(x+y)(x-y)}$$

then I notice that $(x+y)$ shows up **twice** in the integrand. So I could simplify a lot of pieces of the problem down to just u if I set $\boxed{u = x+y.}$ That leaves me with

$$ue^{u(x-y)}$$

in the integral, so the natural choice of v is $\boxed{v = x-y,}$ which turns the integrand into ue^{uv}.

2 I already know how to find u & v in terms of x & y, so now I want to know how to find x & y in terms of u & v. In other words, I need to solve this system for x and y:

$$u = x + y \quad (1)$$
$$v = x - y \quad (2)$$

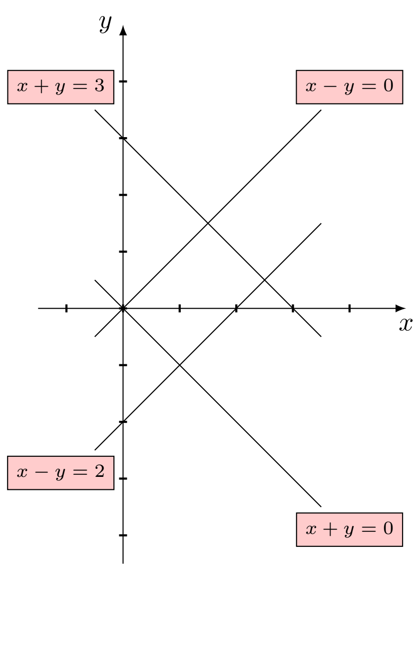
Adding (1)+(2) gives me $u+v=2x$, so

$$\boxed{x = \frac{1}{2}(u+v),}$$

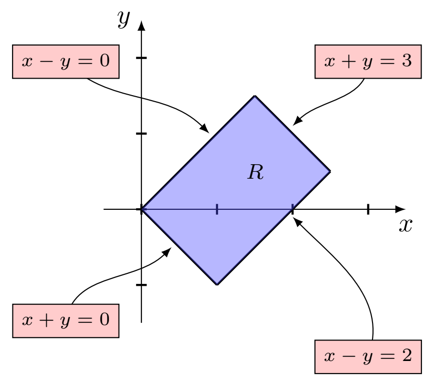
and subtracting (1)–(2) gives me $u-v=2y$, so

$$\boxed{y = \frac{1}{2}(u-v).}$$

3(a) In order to find S, I first need to understand R. Once again, I'll draw a picture since that's the easiest way for me to analyze the information that I have. The problem describes R based on 4 lines, so the first thing I'm going to do to make a picture of R is draw those lines:



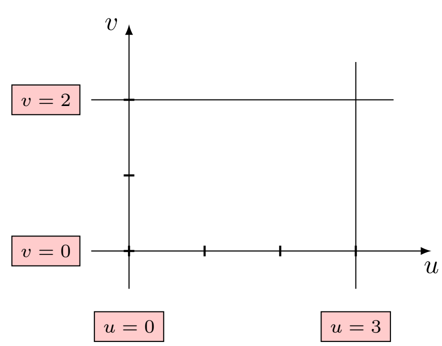
The problem statement tells me that R is the rectangle enclosed by those lines:



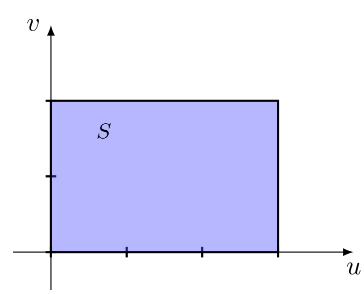
Since I know how to turn x & y into u & v, I can now apply the transformation to each of the sides of R in order to get the sides of S.

x, y	u, v
$x - y = 0$	$v = 0$
$x - y = 2$	$v = 2$
$x + y = 0$	$u = 0$
$x + y = 3$	$u = 3$

Now I can draw the sides of S...

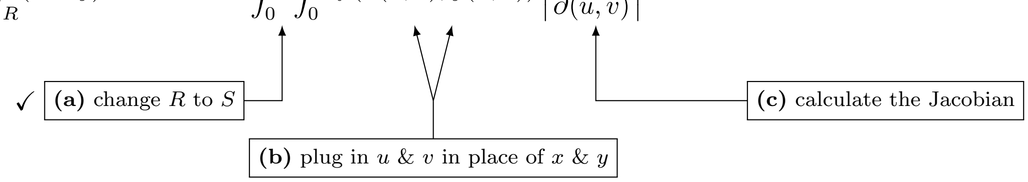


...and fill in the rest.



With the hardest part out of the way, and I can fill in part of the integral:

$$\iint_R (x+y)e^{x^2-y^2}\, dA = \int_0^3 \int_0^2 f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du$$



3(b) I already saw how the integrand is transformed back when I was coming up with the transformation to use:

$$(x+y)e^{x^2-y^2} = (x+y)e^{x^2-y^2} = (x+y)e^{(x+y)(x-y)} = ue^{uv}.$$

Thus

$$\iint_R (x+y)e^{x^2-y^2}\,dA = \int_0^3 \int_0^2 ue^{uv} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

✓ **(a)** change R to S

✓ **(b)** plug in u & v in place of x & y

(c) calculate the Jacobian

3(c) I know what x & y are as functions of u & v, so I can take the derivatives and plug them in to get the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

This is the last part of the integral that I need to fill in:

$$\iint_R (x+y)e^{x^2-y^2}\,dA = \int_0^3 \int_0^2 ue^{uv} \left|-\frac{1}{2}\right| dv\,du$$

✓ **(a)** change R to S

✓ **(b)** plug in u & v in place of x & y

(c) calculate the Jacobian ✓

Now that I'm done with the change of variables, I can evaluate the integral in the usual way:

$$\begin{aligned}
\iint_R (x+y)e^{x^2-y^2}\,dA &= \int_0^2 \int_0^3 ue^{uv} \left|-\frac{1}{2}\right| dv\,du \\
&= \frac{1}{2}\int_0^3 \int_0^2 ue^{uv}\,dv\,du \\
&= \frac{1}{2}\int_0^3 \Big[e^{uv}\Big]_{v=0}^{v=2} du \\
&= \frac{1}{2}\int_0^3 (e^{2u}-1)\,du \\
&= \frac{1}{2}\left[\frac{1}{2}e^{2u} - u\right]_0^3 \\
&= \frac{1}{2}\left(\frac{1}{2}e^6 - 3 - \frac{1}{2} + 0\right) \\
&= \boxed{\frac{e^6-7}{4}}
\end{aligned}$$