

STRONG REDUCTIONS BETWEEN RELATIVES OF THE STABLE RAMSEY'S THEOREM

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ABSTRACT. A complete analysis is given of the computable reductions that hold between SRT_2^2 , SPT_2^2 , and SIPT_2^2 . In particular, while $\text{D}_2^2 \leq_{\text{sW}} \text{SIPT}_2^2 \leq_{\text{sW}} \text{SPT}_2^2 \leq_{\text{sW}} \text{SRT}_2^2$, it is shown that $\text{SRT}_2^2 \not\leq_{\text{sc}} \text{SPT}_2^2 \not\leq_{\text{sc}} \text{SIPT}_2^2 \not\leq_{\text{sc}} \text{D}_2^2$.

1. INTRODUCTION

Much recent work in reverse mathematics has centered on combinatorial principles related to Ramsey's theorem. This paper will examine four such principles and will make a detailed analysis of their logical relationships. All of these principles involve colorings, for which we provide the following definitions.

Definition 1.1.

- If $S \subseteq \omega$, then $[S]^2$ denotes the set of all 2-element subsets of S .
- A *2-coloring of pairs* is a function $f : [\omega]^2 \rightarrow 2$. We hereafter write $f(x, y)$ for $f(\{x, y\})$ with $x < y$, and “coloring” should be understood to mean “2-coloring of pairs.” A 2-coloring of pairs is *stable* if $\lim_u f(x, u)$ exists for all $x \in \omega$.

Given a coloring, we are interested in the existence of sets homogeneous for the coloring in the following senses.

Definition 1.2.

- An infinite set $H \subseteq \omega$ is *homogeneous* for a coloring f if $f \upharpoonright [H]^2$ is constant.
- An infinite set $H = H_L \oplus H_R$ is *p-homogeneous* for a coloring f if $f \upharpoonright H_L \times H_R$ is constant, and is *increasing p-homogeneous* if $f \upharpoonright \{\{x, y\} \mid x < y, x \in H_L, y \in H_R\}$ is constant.
- An infinite set H is *limit homogeneous* for a coloring f if $\lim_u f(\cdot, u)$ is total and constant on H .

With these definitions in hand, we state and name the following principles, in whose relative strength we are interested.

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Definition 1.3.

- SRT_2^2 is the statement that for every stable 2-coloring of pairs $f : [\omega]^2 \rightarrow 2$ there exists a set H homogeneous for f .
- SPT_2^2 is the statement that for every stable 2-coloring of pairs $f : [\omega]^2 \rightarrow 2$ there exists a set H p -homogeneous for f .
- SIPT_2^2 is the statement that for every stable 2-coloring of pairs $f : [\omega]^2 \rightarrow 2$ there exists a set H increasing p -homogeneous for f .
- D_2^2 is the statement that for every stable 2-coloring of pairs $f : [\omega]^2 \rightarrow 2$ there exists a set H limit homogeneous for f .

SPT_2^2 and SIPT_2^2 were first studied in the context of combinatorics by Erdős and Rado in [8] and in the context of reverse mathematics by Dzhafarov and Hirst in [5]. Our interest will be in studying these principles in terms of computable and Weihrauch reductions, to whose definition we now turn. The principles SRT_2^2 , SPT_2^2 , SIPT_2^2 , and D_2^2 , together with many of the principles encountered in reverse mathematics, take the syntactic form

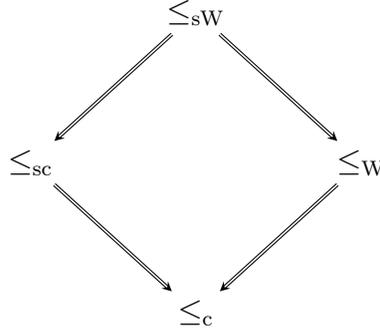
$$\forall X(\Phi(X) \rightarrow \exists Y\Psi(X, Y)),$$

where Φ and Ψ are arithmetical predicates. We call principles of this form *problems*, and given a problem P we call objects X such that $\Phi(X)$ holds *instances* of the problem and objects Y such that $\Psi(X, Y)$ holds *solutions* to the instance X of the problem P . Given this language, we define the following notions of reducibility between two problems.

Definition 1.4. Let P and Q be problems.

- P is *computably reducible* to Q , written $P \leq_c Q$, if every instance X of P computes an instance \hat{X} of Q such that whenever \hat{Y} solves \hat{X} , $X \oplus \hat{Y}$ computes a solution Y to X .
- P is *strongly computably reducible* to Q , written $P \leq_{sc} Q$, if every instance X of P computes an instance \hat{X} of Q such that whenever \hat{Y} solves \hat{X} , \hat{Y} computes a solution Y to X .
- P is *Weihrauch reducible* to Q , written $P \leq_W Q$, if there are Turing functionals Φ and Γ such that whenever X is an instance of P , Φ^X is an instance of Q , and whenever \hat{Y} solves Φ^X , $\Gamma^{X \oplus \hat{Y}}$ solves X .
- P is *strongly Weihrauch reducible* to Q , written $P \leq_{sW} Q$, if there are Turing functionals Φ and Γ such that whenever X is an instance of P , Φ^X is an instance of Q , and whenever \hat{Y} solves Φ^X , $\Gamma^{\hat{Y}}$ solves X .

These four notions are related by the following implications and no others [9].



Each of these notions is meant to capture the intuitive idea that if one has the ability to solve \mathbf{Q} , one may in an algorithmic way use this ability to solve \mathbf{P} . While such ideas have been used at least implicitly in the reverse mathematics literature for some time, they were first presented as objects of study quite recently. Weihrauch and strong Weihrauch reducibility were introduced by Weihrauch in [10] in the context of degrees of discontinuity, and in the context of reverse mathematics were independently discovered by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [2]. Computable reducibility was developed by Dzhafarov [4].

Each of the principles SRT_2^2 , SPT_2^2 , SIPT_2^2 , and D_2^2 is equivalent to each of the others over RCA_0 [1]. This paper will analyze the relationships between these principles in terms of the stronger notions of reducibility introduced in Definition 1.4. It is easy to see that $\text{D}_2^2 \leq_{\text{sW}} \text{SIPT}_2^2 \leq_{\text{sW}} \text{SPT}_2^2 \leq_{\text{sW}} \text{SRT}_2^2$. Furthermore, we have

Proposition 1.5. $\text{SRT}_2^2 \leq_W \text{SPT}_2^2 \leq_W \text{SIPT}_2^2$.

We will use the following lemma to prove Proposition 1.5.

Lemma 1.6. Fix $i < 2$. There is a Turing functional Φ such that if $f : [\omega]^2 \rightarrow 2$ is a stable 2-coloring of pairs and L is an infinite set limit homogeneous for f with color i , then $\Phi^{f \oplus L \oplus \{i\}}$ describes an infinite set H homogeneous for f .

Proof of Lemma 1.6. Fix f and L . We may compute from $f \oplus L \oplus \{i\}$ an infinite set $H = \{h_0, h_1, h_2, \dots\}$ homogeneous for f as follows. Let h_0 be the least element of L , h_1 the least element of L such that $\{h_0, h_1\}$ is finite homogeneous for f with color i , and h_{k+1} the least element of L such that $\{h_0, h_1, \dots, h_k\}$ is finite homogeneous for f with color i . The stability of f and the fact that L is limit homogeneous for f with color i imply that this enumeration is computable from f , L , and i ; this together with the ordering of enumeration implies that H is computable in the given data. \square

Proof of Proposition 1.5. Fix a stable 2-coloring of pairs $f : [\omega]^2 \rightarrow 2$ and an infinite set $I = I_0 \oplus I_1$ increasing p -homogeneous for f . Let i_0 be the least element of I_0 and i_1 the least element of I_1 such that $i_0 < i_1$. Because f is stable and I is increasing p -homogeneous for f , I_0 is limit homogeneous for f with color $f(i_0, i_1)$. Since $f(i_0, i_1)$ and I_0 are computable from $f \oplus I$, by the lemma $f \oplus I$ computes an infinite set H homogeneous for f , hence the infinite set $H \oplus H$ p -homogeneous for f . \square

By contrast to Proposition 1.5, Dzhafarov [3] showed that $\text{SRT}_2^2 \not\leq_W \text{D}_2^2$ and $\text{SRT}_2^2 \not\leq_{\text{sc}} \text{D}_2^2$. We strengthen the latter result by replacing SRT_2^2 with SIPT_2^2 as follows.

Theorem 1.7. $\text{SIPT}_2^2 \not\leq_{\text{sc}} \text{D}_2^2$.

This follows by a straightforward adaptation of the proof of Corollary 3.6 in [3], but in addition we provide the following much simpler proof.

Proof of Theorem 1.7. We will choose $f : [\omega]^2 \rightarrow 2$ be a non-computable instance of SIPT_2^2 all of whose solutions compute \emptyset' . To that end, fix a c.e. approximation of \emptyset' , $\{X_s\}_{s \in \omega}$, with least modulus μ . Now define

$$f(x, y) = \begin{cases} 0, & \text{if } y - x \leq \max\{\mu(z) : z \leq x\}; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\lim_u f(x, u) = 1$ for all $x \in \omega$. Let $H = H_L \oplus H_R$ be any solution to f (so H is increasing p -homogeneous for f with color 1). Then if $z \in \omega$, we can compute from H whether $z \in \emptyset'$ as follows: find the least $x \in H_L$ with $z \leq x$, and the least $y > x$ with $y \in H_R$. Since $f(x, y) = 1$, $y - x > \mu(z)$. Thus $z \in \emptyset'$ if and only if $z \in X_{y-x}$.

So every solution to f computes \emptyset' . But every instance of D_2^2 —in particular, every instance computable from f —has a solution which does not compute \emptyset' ([6]), hence does not compute any infinite set increasing p -homogeneous for f . \square

Our focus in this paper is on the following two theorems, which together with Theorem 1.7 show that none of the implications in Proposition 1.5 can be strengthened to strong computable reductions.

Theorem 1.8 (First Main Theorem). $\text{SRT}_2^2 \not\leq_{\text{sc}} \text{SPT}_2^2$.

Theorem 1.9 (Second Main Theorem). $\text{SPT}_2^2 \not\leq_{\text{sc}} \text{SIPT}_2^2$.

The proofs of the two main theorems will appear later on in their own sections. In section 2 we will develop the combinatorial tool that will be central to those proofs. In section 3 we will prove the first main theorem, and in section 4 we will prove the second main theorem. Section 5 summarizes the results of this paper and, by combining them with the two results of [3] cited above, presents a complete analysis of D_2^2 , SIPT_2^2 , SPT_2^2 , and SRT_2^2 in terms of computable and Weihrauch reductions.

2. TREE LABELING

To prove the main theorems we will use a pair of *tree labeling* arguments. The tree labeling method was first introduced by Dzhafarov in [3] to prove that $\text{COH} \not\leq_{\text{sc}} \text{SRT}_2^2$, and has since been used by Dzhafarov, Patey, Solomon, and Westrick in [7] to prove that $\text{COH} \not\leq_{\text{sc}} \text{SRT}_{<\infty}^2$ and that, for $k > l$, $\text{RT}_k^1 \not\leq_{\text{sc}} \text{SRT}_l^2$. The tree labeling arguments of this paper elaborate on the previous methods and make use of the following definitions.

Definition 2.1.

- If α is a nonempty string of natural numbers, then $\alpha^\#$ denotes $\alpha \upharpoonright |\alpha| - 1$.
- If α and β are strings of natural numbers, then $\alpha * \beta$ denotes α concatenated by β .
- If α is a string of natural numbers and $x \in \omega$, then $\alpha * x$ denotes $\alpha * \langle x \rangle$.
- If $\alpha, \alpha * x \in T$ for some tree T , we say that $\alpha * x$ is a *successor* of α in T .
- If $A, B \subseteq \omega$, we write $A < B$ if every element of A is less than every element of B .

We now proceed define *tree labeling with two labels*.

Definition 2.2. Let $H \subset \omega$ be finite and $I \subset \omega$ be infinite, with $H < I$. Let Γ be a Turing functional. Finally, let $k \in \omega$. We define the tree $T(k, \Gamma, H, I) \subseteq I^{<\omega}$ by $\emptyset \in T(k, \Gamma, H, I)$ and for a nonempty string α , $\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \text{ran}(\alpha^\#)$ and no $b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = 1.$$

Here the finite sets F_L, F_R represent additions to be made to the left and right columns, respectively, of a p -homogeneous set in our later constructions while H represents the initial segment of a p -homogeneous set built by some stage of a construction.

Remark 2.3. $T = T(k, \Gamma, H, I)$ has the following three properties:

- If T is not well-founded and P is any infinite path through T , then $\text{ran}(P) \subseteq I$ is infinite and for all $F_L, F_R \subseteq \text{ran}(P)$ and all $b > a \geq k$,

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \simeq \Gamma^{H \cup (F_L \oplus F_R)}(b) \simeq 0.$$

- If $\alpha \in T$, then if α is not terminal and $\text{ran}(\alpha) < x \in I$, $\alpha * x \in T$.
- If $\alpha \in T$ is terminal, then

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = 1$$

for some $F_L, F_R \subseteq \text{ran}(\alpha)$ and some $b > a \geq k$.

Definition 2.4. When $T = T(k, \Gamma, H, I)$ is well-founded, we *label* the nodes of T recursively, starting at the terminal nodes. Each node is labeled with an ordered pair whose elements may be natural numbers or the symbol ∞ .

- If $\alpha \in T$ is terminal, we label α with the least pair $\langle a, b \rangle$ of elements $b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = 1$$

for some $F_L, F_R \subseteq \text{ran}(\alpha^\#)$.

- If $\alpha \in T$ is not terminal, then we determine the label of α starting with the second element as follows:
 - If there is any $b \in \omega$ such that infinitely many of the successors of α have labels with second element b , then we let the least such b be the second element of the label of α . Otherwise, we let ∞ be the second element of the label of α .
 - Now suppose the second element b of the label of α has been determined already; we will determine its first element a according as $b \in \omega$ or $b = \infty$. If $b = \infty$, then we let a be the least finite number appearing as the first element of the label of infinitely many successors of α , or else if there is no such finite number we let $a = \infty$. If $b \in \omega$, we restrict our attention to just those successors of α whose labels' second element is b , and let a be the least number appearing as the first element of the label of infinitely many successors of α . Observe that $a < b$.

Note that no label which has the symbol ∞ as its first element has a finite number as its second.

Definition 2.5. Suppose $T = T(k, \Gamma, H, I)$ is well-founded. Then the *labeled subtree* $T^L = T^L(k, \Gamma, H, I)$ of T is obtained from T as follows. First, the root node of T (namely \emptyset) is added to T^L . Now suppose we have added to T^L some non-terminal node α of T . We then add to T^L some of the successors of α , thus:

- If α has label $\langle a, b \rangle \in \omega^2$ in T , then we add to T^L all those successors of α with the same label.
- If α has label $\langle a, \infty \rangle$ for some $a \in \omega$ in T , then if infinitely many successors of α have label $\langle a, b \rangle \in \omega^2$, then for each $b \in \omega$ such that $\langle a, b \rangle$ appears as the label of a successor of α , we select the least x such that $\alpha * x$ has that label

and add $\alpha * x$ to T^L ; and if on the other hand cofinitely many successors of α have the same label (namely $\langle a, \infty \rangle$), we add all such successors to T^L .

- Otherwise, if α has label $\langle \infty, \infty \rangle$, then if infinitely many successors of α have label $\langle a, b \rangle \in \omega^2$, then for each such pair that appears as the label of a successor of α , we select the least x such that $\alpha * x$ has that label and add $\alpha * x$ to T^L ; and if cofinitely many successors of α have label $\langle a, \infty \rangle$ for $a \in \omega$, then for each $a \in \omega$ such that $\langle a, \infty \rangle$ appears as the label of a successor of α , we select the least x such that $\alpha * x$ has that label and add $\alpha * x$ to T^L ; and otherwise if cofinitely many successors of α have label $\langle \infty, \infty \rangle$, we add those successors to T^L .

All the nodes of T^L retain the labels they had as nodes of T . Note that every node terminal in T is also terminal in T^L , and that every non-terminal node in both T and T^L has infinitely many successors.

Definition 2.6. A node $\alpha \in T^L$ is called a *transition node* if the symbol ∞ appears in the label of α , and appears strictly fewer times in the label of each successor of α .

3. FIRST MAIN THEOREM

Theorem 3.1. *There exists a stable 2-coloring of pairs $f : [\omega]^2 \rightarrow 2$ and a family Y of infinite sets such that no $(f \oplus P)$ -computable set is homogeneous for f for any $P \in Y$, and every stable 2-coloring of pairs $f' : [\omega]^2 \rightarrow 2$ computable from f has either an $(f \oplus P)$ -computable p -homogeneous set for some $P \in Y$, or if not then some p -homogeneous set which does not compute a set homogeneous for f .*

Before proving Theorem 3.1, we observe how the First Main Theorem is a direct consequence. Let f and Y be as in the statement of Theorem 3.1, let Φ, Ψ be any Turing functionals, and suppose that Φ^f is a stable 2-coloring of pairs. Let H be set p -homogeneous for Φ^f . Then by Theorem 3.1, Ψ^H is not homogeneous for f . Thus $\text{SRT}_2^2 \not\leq_{\text{sc}} \text{SPT}_2^2$.

We define the following notion of forcing which we shall need to prove each of the main theorems.

Definition 3.2. Let \mathbb{C} denote the following notion of forcing. A condition is an ordered triple $p = \langle \sigma^p, l^p, |p| \rangle$ where $|p| \in \omega$, $\sigma^p : [|p|]^2 \rightarrow 2$, $l^p : |p| \rightarrow 2 \times \omega$, and $l^p(x) = \langle i, z \rangle$ implies that if $\sigma^p(x, y)$ is defined and $y \geq z$ then $\sigma^p(x, y) = i$.

From any sufficiently generic filter \mathcal{G} for \mathbb{C} we obtain a stable 2-coloring of pairs $f = \bigcup_{p \in \mathcal{G}} \sigma^p : [\omega]^2 \rightarrow 2$ together with a function $l = \bigcup_{p \in \mathcal{G}} l^p : \omega \rightarrow \omega \times 2$ such that for each $x \in \omega$, $\lim_u f(x, u) = (l(x))_0$.

Lemma 3.3. If $p_0 \geq p_1 \geq p_2 \geq \dots$ is a sequence of \mathbb{C} -conditions which is 3-generic relative to some set $P \subseteq \omega$, and if $f = \bigcup_s \sigma^{p_s} : [\omega]^2 \rightarrow 2$, then $f \oplus P$ does not compute a homogeneous set for f .

Proof of Lemma 3.3. Fix $P \subseteq \omega$ such that $p_0 \geq p_1 \geq p_2 \geq \dots$ is 3-generic with respect to P and fix a Turing functional Γ . Let $W_{P, \Gamma}$ be the set of all conditions p which force one of the following two statements:

- (a) $\Gamma^{f \oplus P}$ does not define an infinite set;
- (b) there are $x, y \in \omega$ with $\Gamma^{f \oplus P}(x) \downarrow = \Gamma^{f \oplus P}(y) \downarrow = 1$ and $\lim_u f(x, u) \neq \lim_u f(y, u)$.

$W_{P,\Gamma}$ is Σ_3^0 -definable in P , and we claim that $W_{P,\Gamma}$ is dense in \mathbb{C} . To see this, let p be any condition none of whose extensions force (a), and suppose by way of contradiction that no extension of p forces (b). This means that for every $x, y \in \omega$, if $q \leq p$ then q does not force both that $\lim_u f(x, u) \neq \lim_u f(y, u)$ and that $\Gamma^{f \oplus P}(x) \downarrow = \Gamma^{f \oplus P}(y) \downarrow = 1$. Then in particular there are no $x, y \geq |p|$, τ extending σ^p , and L extending l^p such that

- (1) τ respects L ;
- (2) $\Gamma^{\tau \oplus P \upharpoonright |p|}(x) \downarrow = \Gamma^{\tau \oplus P \upharpoonright |p|}(y) \downarrow = 1$; and
- (3) $(L(x))_0 \neq (L(y))_0$.

Now if there is no such x, y, τ, L satisfying (1) and (2), then p forces that $\Gamma^{f \oplus P}$ does not define an infinite set, which is a contradiction. Therefore there are x, y, τ, L satisfying (1) and (2). But (3) is independent of (1) and (2); given x, y, τ satisfying (1) and (2) we may find L extending l^p and compatible with τ such that $(L(x))_0 \neq (L(y))_0$. We conclude that W is dense in \mathbb{C} . \square

Proof of Theorem 3.1. We build

- a sequence of \mathbb{C} -conditions $p_0 \geq p_1 \geq p_2 \geq \dots$ with $\lim_s |p_s| = \infty$;
- sequences of finite sets (initial segments of p -homogeneous sets) $H_{j,0}^\Phi \subseteq H_{j,1}^\Phi \subseteq H_{j,2}^\Phi \subseteq \dots$ for each Turing functional Φ and each $j < 2$;
- a sequence of infinite sets (reservoirs) $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ with $H_{j,s}^\Phi < I_s$ for each Φ, j, s ;
- a sequence of finite families $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ of infinite subsets of ω ;

and we define $f = \bigcup_s \sigma^{p_s}$, $H_j^\Phi = \bigcup_s H_{j,s}^\Phi$ for each $j < 2$, and $Y = \bigcup_s Y_s$. The construction will ensure the following requirements, for all $i \in \omega$ and Turing functionals Φ, Γ, Δ :

- \mathcal{P}_i : the sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ is 3-generic relative to each $P \in Y_i$;
- $\mathcal{Q}_{\Phi,i}$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$ -computable p -homogeneous set for some $P \in Y$ or else both H_0^Φ and H_1^Φ are infinite in both columns (in other words, given our encoding of p -homogeneous sets, contain infinitely many even and infinitely many odd numbers);
- $\mathcal{R}_{\Gamma,\Delta}^\Phi$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$ -computable p -homogeneous set for some $P \in Y$; or else if $\Gamma^{H_0^\Phi}$ defines an infinite set then this set is not homogeneous for f ; or else if $\Delta^{H_1^\Phi}$ defines an infinite set then this set is not homogeneous for f .

By way of explaining the \mathcal{Q} and \mathcal{R} requirements, note that by Lemma 3.3, if there is any sequence satisfying the \mathcal{P} requirements and such that for some $P \in Y$ $f \oplus P$ computes a set p -homogeneous for Φ^f , there will then be a set which is p -homogeneous for Φ^f but which computes no set homogeneous for f .

Construction.

Devote infinitely many stages $s \in \omega$ to each requirement. Let p_0 be any condition with $|p_0| = 0$. For each Φ let $H_{0,0}^\Phi = H_{1,0}^\Phi = \emptyset$, and let $I_0 = \omega$ and $Y_0 = \emptyset$. At stage

$s + 1$ assume by way of induction that we have $p_s, H_{j,s}^\Phi$ for $j < 2$ and all Φ, I_s , and Y_s and assume that if $H_{j,s}^\Phi$ is nonempty for some j and Φ , then p_s forces that Φ^f is a stable coloring of pairs and that $\Phi^f(x, y) = j$ whenever $2x, 2y + 1 \in H_{j,s}^\Phi$ or when $y \in I_s$ and either $2x \in H_{j,s}^\Phi$ or $2x + 1 \in H_{j,s}^\Phi$. At the end of a stage any of $p_{s+1}, H_{j,s+1}^\Phi, I_{s+1}$, or Y_{s+1} not yet defined should be taken to be identical to $p_s, H_{j,s}^\Phi, I_s$, or Y_s , respectively.

\mathcal{P} requirements. Suppose s is dedicated to requirement \mathcal{P}_i for some $i < s$ and that it is the $\langle n, m \rangle^{\text{th}}$ such stage. If $n > |Y_i|$ do nothing. Otherwise, let P be the n^{th} member of the family Y_i in some fixed enumeration and let W be the m^{th} $\Sigma_3^0(P)$ set in some fixed enumeration. If p_s has an extension q in W , fix q and let $p_{s+1} = q$, so that $p_0 \geq p_1 \geq \dots \geq p_s \geq p_{s+1} \geq \dots$ meets W . Otherwise, do nothing, and $p_0 \geq p_1 \geq \dots \geq p_s \geq p_{s+1} \geq \dots$ avoids W .

\mathcal{Q} requirements. Suppose s is dedicated to $\mathcal{Q}_{\Phi,i}$. Without loss of generality we assume that p_s decides whether or not Φ^f is a stable 2-coloring of pairs. If p_s forces that Φ^f is not such a coloring, do nothing. Otherwise, we consider two cases.

- If for some $j < 2$ and $k \in \omega$ there is no extension of p_s which forces that $\lim_u \Phi^f(x, u) = j$ for some $x \geq k$ in I_s , then $P = \{x \in I_s : x \geq k\}$ is limit homogeneous for Φ^f with color $1 - j$, and so $(f \oplus P)$ computes a set p -homogeneous for Φ^f . We set $Y_{s+1} = Y_s \cup \{P\}$. This satisfies the requirement (since, as remarked earlier, this means there is a set which is p -homogeneous for Φ^f but which computes no set homogeneous for f).
- If no such j, k exist, then there are numbers $x_{00}, x_{01}, x_{10}, x_{11} \in I_s$ and an extension of p_s forcing that $H_{j,s}^\Phi \cup \{2x_{j0}, 2x_{j1} + 1\}$ is finite p -homogeneous for Φ^f and $\lim_u \Phi^f(x_{ji}, u) = j$ for each $i, j < 2$. In this case let p_{s+1} be such an extension of p_s , let $H_{j,s+1}^\Phi = H_{j,s}^\Phi \cup \{2x_{j0}, 2x_{j1} + 1\}$, and let $I_{s+1} = \{x : m < x \in I_s\}$ where m is greater than the stabilization points under Φ^f of every element of $H_{0,s+1}^\Phi \cup H_{1,s+1}^\Phi$. Observe that both columns have been extended by one element.

\mathcal{R} requirements. Suppose s is dedicated to $\mathcal{R}_{\Gamma,\Delta}^\Phi$ and assume that p_s forces that Φ^f is a stable coloring of pairs. The goal of this requirement is to extend by some finite set either the initial segment of H_0^Φ or the initial segment of H_1^Φ that we have constructed so far, subject to the following condition: if we extend H_0^Φ , then Γ does not compute a homogeneous set for f from any further extension of H_0^Φ ; and if on the other hand we extend H_1^Φ , then Δ does not compute a homogeneous set for f from any further extension of H_1^Φ . When we have so extended one of H_0^Φ, H_1^Φ , we will say that we have successfully *diagonalized* against such computations.

We intend to accomplish this diagonalization in the following way. There will be two numbers, say a and b , and two finite sets F_L and F_R , all arising from a tree labeling construction, about which we know either that

$$\Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(b) \downarrow = 1$$

or else that

$$\Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(a) \downarrow = \Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(b) \downarrow = 1.$$

Let us suppose we know the first. In other words, we know that if we extend H_0^Φ by $F_L \oplus F_R$, then the set computed from $H_0^\Phi \cup (F_L \oplus F_R)$ by Γ will contain a and b . Thus to diagonalize—i.e. to ensure that the set so computed is not homogeneous

for f —we will choose an extension q of the condition p_s such that $\sigma^q(a, b)$, $(l^q(a))_0$, and $(l^q(b))_0$ are not all equal. This will guarantee that no set containing both a and b is homogeneous for the coloring eventually obtained by extending σ^q . The tension of the proof arises from the fact that, while diagonalizing in this way, we also need q to force that elements of F_L and F_R have the right limits under Φ^f so that $H_0^\Phi \cup (F_L \oplus F_R)$ can in fact be extended to a p -homogeneous set.

The construction divides into two cases, **Case B** and **Case A**. In Case B we suppose there is an $i < 2$, a condition $q \leq p_s$, an infinite set $I \subseteq I_s$, and a set $Q = \{\langle x, a_x, b_x \rangle : x \in I, a_x, b_x \in \omega\}$ such that there is exactly one triple in Q with first element x for each $x \in I$, $a_x < b_x$, and such that infinitely many numbers $b_x \in \omega$ appear as third elements of triples in Q ; and we suppose these have the following property: for any $x \in I$ and any extension r of q such that $\sigma^r(a_x, b_x)$, $(l^r(a_x))_0$, and $(l^r(b_x))_0$ are not all equal will force that $\lim_u \Phi^f(x, u) = i$. In Case A we suppose this is not so. The reason for choosing this way of dividing the cases will become apparent later. Case B may be regarded as the difficult case, where the real heavy lifting of the proof takes place; Case A may be regarded as the relatively easy case.

Case A. Let $T_0 = T(|p_s|, \Gamma, H_{0,s}^\Phi, I_s)$. If T_0 is not well-founded then let I_{s+1} be the range of an infinite path through T_0 . Observe that in this case the requirement is satisfied. If T_0 is well-founded, then let T_0^L be the labeled subtree of T_0 .

We now try to define two sequences, conditions

$$p_s \geq q_0 \geq q_1 \geq q_2 \geq \dots$$

and nodes of T_0^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \dots$$

where for all $j \geq 0$, α_{j+1} is a successor of α_j and for all $j \geq 0$ the condition q_j forces that

$$\lim_u \Phi^f(x, u) = 0$$

for all $x \in \text{ran}(\alpha_j)$. We begin the definition of these sequences as follows.

- If $\emptyset = \alpha_0$ has label $\langle a, b \rangle$ with $a, b \in \omega$, let q_0 be any extension of p_s having $\sigma^{q_0}(a, b)$, $(l^{q_0}(a))_0$, and $(l^{q_0}(b))_0$ not all equal. In this case the diagonalization for the present requirement is now complete.
- Otherwise, let $q_0 = p_s$.

We then proceed by induction until either the induction fails and we satisfy the requirement by adding a certain set P to Y , or else the induction always succeeds at every non-terminal node. In the latter case, once we reach a terminal node we will be ready to diagonalize. Suppose we have defined q_n and α_n and that the latter is not terminal in T_0^L . Recall that q_n forces that there is some $m \in \omega$ such that $\Phi^f(x, y) = 0$ for $x \in \text{ran}(\alpha_n)$ and $y \geq m$. Let $S = S(n)$ be the set of all successors $\alpha_n * x$ of α_n with $x \geq m$. The induction breaks into cases according as α_n is or is not a transition node. At the first suitable transition node, we set up to diagonalize.

Case A.1. If α_n is not a transition node, let $P = \{x : \alpha_n * x \in S\}$. We look for an $x^* \in P$ and an extension q of q_n which forces that

$$\lim_u \Phi^f(x^*, u) = 0.$$

If we find such, we let $q_{n+1} = q$ and we let α_{n+1} be any $\beta \in S$ with $\beta(n) = x^*$; if we find no such, then P is limit-homogeneous for Φ^f and thus $f \oplus P$ computes a p -homogeneous set for Φ^f (for example, we can thin P computably in f to a set G homogeneous for Φ^f ; then $G \oplus G$ is p -homogeneous for Φ^f). In this case we set $Y_{s+1} = Y_s \cup \{P\}$ and $p_{s+1} = q_n$, satisfying the requirement and ending stage s .

Case A.2. If α_n is a transition node, the induction breaks into two cases. The header for each case gives a shorthand for the sort of transition being discussed.

Case A.2.1. $\langle \infty, \infty \rangle \rightarrow \langle a, \infty \rangle$

If α_n has label $\langle \infty, \infty \rangle$ and every successor of α_n has a label in which the symbol ∞ appears exactly once, then we proceed as in the non-transition case.

Case A.2.2. $\langle \infty, \infty \rangle \rightarrow \langle a, b \rangle$ or $\langle a, \infty \rangle \rightarrow \langle a, b \rangle$

If α_n has a label in which the symbol ∞ appears but every successor of α_n is labeled only with finite numbers, then let

$$P = \{\langle x, a, b \rangle : \alpha_n * x \in S \text{ and } \alpha_n * x \text{ has label } \langle a, b \rangle \text{ and } b > |q_n|\}.$$

We look for a tuple $\langle x^*, a^*, b^* \rangle \in P$ and an extension q of q_n which forces that

$$\lim_u \Phi^f(x^*, u) = 0$$

and is such that $\sigma^q(a^*, b^*)$, $(l^q(a^*))_0$, and $(l^q(b^*))_0$ are not all equal. Observe that, by the assumption that we are not in Case B, we must find such a tuple and extension. For otherwise fix Q to be a set of triples $\langle x, a_x, b_x \rangle$ such that for every b with $\langle y, a, b \rangle \in P$ for some y and a , there is precisely one triple $\langle x, a_x, b_x \rangle$ in Q with $b_x = b$ and let $I = \{x : \exists a_x, b_x (\langle x, a_x, b_x \rangle \in Q)\}$. Then we are in Case B with $i = 1$ and $q = q_n$. Thus, select such an extension q and let $q_{n+1} = q$ and we let α_{n+1} be any $\beta \in S$ with $\beta(n) = x^*$ and label $\langle a^*, b^* \rangle$. As α_{n+1} is not a transition node, we return to Case A.1.

Case B. Fix an $i < 2$, a condition $q \leq p_s$, an infinite set $I \subseteq I_s$, and a set $Q = \{\langle x, a_x, b_x \rangle : x \in I, a_x, b_x \in \omega\}$ such that there is exactly one triple in Q with first element x for each $x \in I$; such that infinitely many numbers $b_x \in \omega$ appear as third elements of triples in Q ; and such that for any $x \in I$ and any extension r of q such that $\sigma^r(a_x, b_x)$, $(l^r(a_x))_0$, and $(l^r(b_x))_0$ are not all equal will force that $\lim_u \Phi^f(x, u) = i$. Suppose without loss of generality that $i = 1$. In the tree labeling construction which follows, we will extend q several times. As we gradually extend the forcing condition, for any given x it might occur—and we may as well assume—that the limiting color of a_x will be forced, and that the stabilization point will fall below b_x so that the color of (a_x, b_x) will be forced also. However, since there are infinitely many different numbers b_x , by restricting our attention to those x 's for which the limiting color of b_x has not yet been forced, we will always have the option of choosing an extension q as above, i.e. such that $\sigma^q(a_x, b_x)$, $(l^q(a_x))_0$, and $(l^q(b_x))_0$ are not all equal, and thus force that $\lim_u \Phi^f(x, u) = 1$. The fact that we can always accomplish this while making progress toward satisfying the present requirement underlies the remainder of the construction, and so we find it useful to define a term describing extensions q_n which accomplish this.

Definition 3.4. A forcing extension q is said to *press the button of x* or to *press b_x* if $\sigma^q(a_x, b_x)$, $(l^q(a_x))_0$, and $(l^q(b_x))_0$ are not all equal.

Thus (assuming as above that $i = 1$) any forcing extension which presses the button of x forces that $\lim_u \Phi^f(x, u) = 1$.

For this case we need to modify our definition of tree labeling. We will use *tree labeling with three labels*. The definition of $T(k, \Gamma, H, I)$ is for this case changed to the following: $\emptyset \in T(k, \Gamma, H, I)$ and for a nonempty string α , $\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \text{ran}(\alpha^\#)$ and no $c > b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(a) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(b) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(c) \downarrow = 1.$$

The method for labeling the nodes of $T(k, \Gamma, H, I)$ extends the method from tree labeling with two labels in the natural way, as does the method for selecting the nodes of the labeled subtree $T^L(k, \Gamma, H, I)$. Remark 2.3 applies *mutatis mutandis*. Additionally, we revise the definition of a transition node: a node $\alpha \in T^L$ is called a transition node if the symbol ∞ appears in the label of α , and appears strictly fewer times *and no more than once* in the label of each successor of α .

Now let $T_1 = T(|q|, \Delta, H_{1,s}^\Phi, I_s)$. If T_1 is not well founded then let I_{s+1} be the range of an infinite path through T_1 . Observe that in this case the requirement is satisfied. If T_1 is well founded, let T_1^L be the labeled subtree of T_1 .

We now try to define two sequences, conditions

$$q \geq q_0 \geq q_1 \geq q_2 \geq \dots$$

and nodes of T_1^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \dots$$

where for all $j \geq 0$, α_{j+1} is a successor of α_j and for all $j \geq 0$ the condition q_j forces that

$$\lim_u \Phi^f(x, u) = 1$$

for all $x \in \text{ran}(\alpha_j)$. We begin the definition of these sequences as follows.

- If $\emptyset = \alpha_0$ has label $\langle a, b, c \rangle$ with $a, b, c \in \omega$, let q_0 be any extension of q having $\sigma^{q_0}(a, b)$, $(l^{q_0}(a))_0$, and $(l^{q_0}(b))_0$ not all equal.
- If $\emptyset = \alpha_0$ has label $\langle a, b, \infty \rangle$ with $a, b \in \omega$, let q_0 be any extension of q having $\sigma^{q_0}(a, b)$, $(l^{q_0}(a))_0$, and $(l^{q_0}(b))_0$ not all equal.
- Otherwise, let $q_0 = q$.

We then proceed by induction. Suppose we have defined q_n and α_n and that the latter is not terminal in T_1^L . Recall that q_n forces that there is some $m \in \omega$ such that $\Phi^f(x, y) = 1$ for $x \in \text{ran}(\alpha_n)$ and $y \geq m$. Let S be the set of all successors $\alpha_n * x$ of α_n with $x \geq m$ and $b_x > |q_n|$. The induction breaks into cases according as α_n is or is not a transition node, with diagonalization occurring at transition nodes.

Case B.1. If α_n is not a transition node, let $P = \{x : \alpha_n * x \in S\}$. We choose any $x^* \in P$ and any extension r of q_n which presses b_{x^*} , and let $q_{n+1} = r$ and $\alpha_{n+1} = \gamma$ for any $\gamma \in S$ having $\gamma(n) = x^*$.

Case B.2. If α_n is a transition node, the induction breaks into four subcases. Each case is detailed in a separate bullet point below, and the header under each bullet point gives a shorthand for the sort of transition being discussed. The real work of the construction, making full use of the triple labeling of T_1^L , takes place in subcases 3 and 4.

Case B.2.1. $\langle \infty, \infty, \infty \rangle \rightarrow \langle a, b, \infty \rangle$ or $\langle a, b, c \rangle$

If α_n has label $\langle \infty, \infty, \infty \rangle$ and every successor of α_n has a label in which the symbol ∞ appears at most once, then we let

$$P = \{x : \alpha_n * x \in S \wedge \forall j < n (a, b > b_{\alpha_n(j)}) \wedge a, b > |q_n|\},$$

where a, b here denote the first two entries in the label of $\alpha * x$ and $b_{\alpha_n(j)}$ denotes the button of $\alpha_n(j)$. We choose any $x^* \in P$ and any extension r of q_n such that, if the label of x^* is $\langle a^*, b^*, c^* \rangle$ with $a^*, b^* \in \omega$ and $c^* \in \omega \cup \{\infty\}$, then r presses b_{x^*} and $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal. Recall the definition of S , whereby we know that $b_x > |q_n|$ for each $x \in P$. Thus for all $x \in P$, l^{q_n} does not commit us to any particular limiting color for b_x . This is why we are free to press the button of x^* . Let $q_{n+1} = r$ and $\alpha_{n+1} = \gamma$ for any $\gamma \in S$ having $\gamma(n) = x^*$ with label $\langle a^*, b^*, c^* \rangle$. As α_{n+1} is not a transition node, we now return to Case B.1.

Case B.2.2. $\langle a, \infty, \infty \rangle \rightarrow \langle a, b, c \rangle$

If α_n has label $\langle a^*, \infty, \infty \rangle$ and every successor of α_n has a label in which only finite numbers appear, then we let

$$P = \{x : \alpha_n * x \in S \wedge \forall j < n (b, c > b_{\alpha_n(j)}) \wedge b, c > |q_n|\},$$

where b, c here denote the second and third entries in the label of $\alpha * x$ and $b_{\alpha_n(j)}$ denotes the button of $\alpha_n(j)$. We choose any $x^* \in P$ and any extension r of q_n that presses b_{x^*} and is such that, if the label of x^* is $\langle a^*, b^*, c^* \rangle$ with $a^*, b^*, c^* \in \omega$, then $\sigma^r(b^*, c^*)$, $(l^r(b^*))_0$, and $(l^r(c^*))_0$ are not all equal. Let $q_{n+1} = r$ and $\alpha_{n+1} = \gamma$ for any $\gamma \in S$ having $\gamma(n) = x^*$ with label $\langle a^*, b^*, c^* \rangle$. As α_{n+1} is not a transition node, we now return to Case B.1.

Case B.2.3. $\langle a, \infty, \infty \rangle \rightarrow \langle a, b, \infty \rangle$

For this case we first define by induction a function **Sort** on nodes $\alpha \in T_1^L$ which records information about how elements that go into $F_L \oplus F_R$ are sorted between F_L and F_R . We define **Sort** as follows:

- If α is terminal and we have labeled α with the triple $\langle a, b, c \rangle$ because

$$\Delta^{H_{1,s}^\Phi \cup (F_L \oplus F_R)}(a) \downarrow = \Delta^{H_{1,s}^\Phi \cup (F_L \oplus F_R)}(b) \downarrow = \Delta^{H_{1,s}^\Phi \cup (F_L \oplus F_R)}(c) \downarrow = 1,$$

then we let $\text{Sort}(\alpha) = \{F_L \oplus F_R\}$.

- If α is not terminal, then we define

$$\text{Sort}(\alpha) = \bigcup \{S : S = \text{Sort}(\beta) \text{ for infinitely many 1-extensions } \beta \text{ of } \alpha\}.$$

Now we define a convenient term. If $\alpha \in T = T(k, \Gamma, H, I)$ is not terminal and if $x, y \in \text{ran}(\alpha)$, we say that x and y *share a column* if there is $F_L \oplus F_R \in \text{Sort}(\alpha)$ such that one of the following is true:

- $x \in F_L$ and $y \in F_L$;

- $x \in F_R$ and $y \in F_R$;
- $x \notin F_L \cup F_R$ or $y \notin F_L \cup F_R$.

Informally, x and y share a column if, in the course of the construction, we can avoid building a p -homogeneous set in which x and y appear in different columns. We now resume the construction.

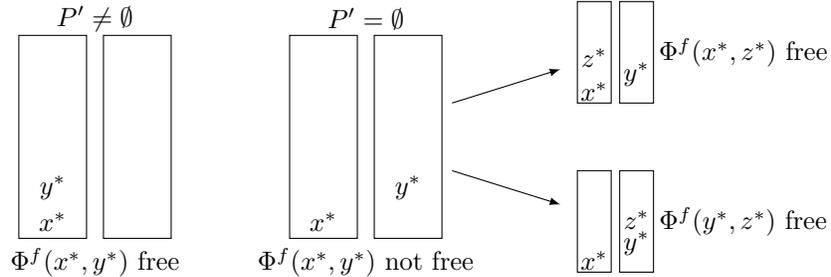
If α_n has label $\langle a^*, \infty, \infty \rangle$ and successor of α_n has a label in which the symbol ∞ appears exactly once, then we let

$$P = \{y : \alpha_n * y \in S \wedge \forall j < n (b > b_{\alpha_n(j)}) \wedge b > |q_n|\},$$

where b here denotes the second entry in the label of $\alpha_n * y$ and $b_{\alpha_n(j)}$ denotes the button of $\alpha_n(j)$. Whether there is much work to be done in this case depends on whether the first node to have label $\langle a^*, \infty, \infty \rangle$ was or was not the root node of T_1^L . Formally, suppose k is the least index such that α_k has a label in which the symbol ∞ appears exactly twice.

- If, on the one hand, $\alpha_k = \alpha_0 = \emptyset$, then we choose any $y^* \in P$ and any extension r of q_n which presses b_{y^*} and is such that, if the label of y^* is $\langle a^*, b^*, \infty \rangle$, then $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal. In this case let $q_{n+1} = r$ and $\alpha_{n+1} = \gamma$ for any $\gamma \in S$ having $\gamma(n) = x^*$ with label $\langle a^*, b^*, \infty \rangle$.
- If, on the other hand, $k > 0$ and $\alpha_k \neq \emptyset$, then there is more work to do. Suppose that $\alpha_k(k-1) = x^*$. Let $P' \subseteq P$ contain precisely the elements y of P which share a column with x^* . This is where we begin to use the triple labels of T_1^L . Either $P' \neq \emptyset$ and we complete the diagonalization in this case, or else $P' = \emptyset$ and we wait until the next case to diagonalize, but we are guaranteed to succeed when we attempt in the next case to find pairs of elements which share a column.

The important idea here is intuitively as follows. Either we may choose y^* from P to share with x^* a column of the p -homogeneous set, or else x^* and y^* do not share a column and then z^* may be chosen at the next transition node to share a column with one of x^* or y^* . In either case, the color that Φ^f assigns to the pair of numbers which share a column can safely be changed without disturbing the construction.



If P' is nonempty, then we choose any $y^* \in P'$ with label $\langle a^*, b^*, \infty \rangle$ and a condition r which extends q_n except possibly having $(l^r(a^*))_1 \neq (l^{q_n}(a^*))_1$ if the latter is defined; and we choose y^* and r such that r presses b_{y^*} and $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal; and we let $q_{n+1} = r$ and $\alpha_{n+1} = \gamma$ for any $\gamma \in S$ having $\gamma(n) = y^*$ with label $\langle a^*, b^*, \infty \rangle$. Finally, if σ is any non-terminal extension of α_{n+1} in T_1^L such that x^* and y^* do not share a column, then we delete

from $T_1^L \sigma$ and all of its extensions. As α_{n+1} is not a transition node, we now return to Case B.1.

Such y^* and r exist in this case for the following reason. Observe that for any $y^* \in P'$ with label $\langle a^*, b^*, \infty \rangle$, $a^* \neq b_{\alpha_n(j)}$ for $j \neq k-1$. Since $b^* > |q_n|$, $\sigma^{q_n}(a^*, b^*)$ is not yet defined, so we may choose r extending q_n with $\sigma^r(a^*, b^*) \neq (l^r(a^*))_0$ —unless $l^{q_n}(a^*)$ is defined and $(l^{q_n}(a^*))_1 \leq b^*$. In this latter case we let $(l^r(a^*))_0 = (l^{q_n}(a^*))_0$ but choose $(l^r(a^*))_1 > b^*$. Note that in this case r is not an extension of q_n but that r does extend p_s . From here we extend r rather than q_n .

To conclude the argument in this case, it remains to observe that changing the stabilization point of a^* as above does not injure our construction in any way. For the colors of pairs of elements of α_n are forced by facts about σ^{q_n} alone. Furthermore, the fact that $\Phi^f(u, y^*) = 1$ for all $u \in \text{ran}(\alpha_n)$, $u \neq x^*$ with parity unequal to that of y^* is settled by facts about the stabilization points of such numbers u , and these are unaffected by altering $(l^{q_n}(a^*))_1$ since a^* is not the button of any such u . This means that choosing $(l^r(a^*))_1 \neq (l^{q_n}(a^*))_1$ can at most interfere with the stabilization point of x^* and keep us from forcing $\Phi^f(x^*, y^*) = 1$. But note that this does not matter for purposes of the p -homogeneous set we are building, since x^* and y^* will not appear in different columns of the p -homogeneous set we are building and so their mutual color is irrelevant.

If on the other hand P' is empty, then we proceed as in the non-transition case and will diagonalize in Case B.2.4 instead.

Case B.2.4. $\langle a, b, \infty \rangle \rightarrow \langle a, b, c \rangle$

If α_n has label $\langle a^*, b^*, \infty \rangle$ and every successor of α_n has a label in which only finite numbers occur, and if we failed to diagonalize at an earlier node in the previous case, then we proceed as follows; otherwise we proceed as in the non-transition case. Let

$$P = \{z : \alpha_n * z \in S \wedge \forall j < n (c > b_{\alpha_n(j)}) \wedge c > |q_n|\},$$

where c here denotes the third entry in the label of $\alpha_n * z$. Suppose k is the least index such that α_k has a label in which the symbol ∞ appears exactly twice and that l is the least index such that α_l has a label in which the symbol ∞ appears exactly once. Suppose that $\alpha_k(k-1) = x^*$ and that $\alpha_l(l-1) = y^*$. Let $P' \subseteq P$ contain precisely those elements of P which share a column with x^* and $P'' \subseteq P$ contain precisely those elements of P which share a column with y^* . At least one of P', P'' must be nonempty; without loss of generality we assume that P'' is nonempty. Then we choose $z^* \in P''$ with label $\langle a^*, b^*, c^* \rangle$ and a condition r which extends q_n except possibly having $(l^r(b^*))_1 \neq (l^{q_n}(b^*))_1$ if the latter is defined; and we choose z^* and r such that $\sigma^r(b^*, c^*)$, $(l^r(b^*))_0$, and $(l^r(c^*))_0$ are not all equal and r presses b_{z^*} . We let $q_{n+1} = r$ and $\alpha_{n+1} = \gamma$ for any $\gamma \in S$ having $\gamma(n) = z^*$ with label $\langle a^*, b^*, c^* \rangle$. Such z^* and r exist by the same reasoning given in the previous case. Finally, if σ is any non-terminal extension of α_{n+1} in T_1^L such that y^* and z^* do not share a column, then we delete from T_1^L σ and all of its extensions. As α_{n+1} is not a transition node, we now return to Case B.1.

We complete stage s as follows. If added some set P to Y , or if we defined I_{s+1} to be the range of an infinite path through T_0 or T_1 , we are done. For if we added some set P to Y , then there will be an $(f \oplus P)$ -computable set p -homogeneous for Φ^f , but by Lemma 3.3 there will be no $(f \oplus P)$ -computable set homogeneous for f . And if we defined I_{s+1} to be the range of an infinite path through T_0 or T_1 —say through T_0 —then from the definition of that tree $\Gamma^{H_0^\Phi}$ does not define an infinite set, let alone one homogeneous for f .

Otherwise, we succeeded either in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_0^L or else in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_1^L ; say we succeeded in defining the sequence of nodes in T_0^L . That tree was in this case well-founded, so for some i , α_n was terminal. Then from the definition of the tree, there are some $F_L, F_R \subseteq \text{ran}(\alpha_n)$ such that $\Gamma^{H_{0,s}^\Phi \cup F_L \oplus F_R}(a) \downarrow = \Gamma^{H_{0,s}^\Phi \cup F_L \oplus F_R}(b) \downarrow = 1$ for some unequal $a, b \geq |p_s|$, say with use u . Let $p_{s+1} = q_n$, $H_{0,s+1}^\Phi = H_{0,s}^\Phi \cup F_L \oplus F_R$, and $I_{s+1} = \{x \in I_s : x > u\}$. \square

4. SECOND MAIN THEOREM

Theorem 4.1. *There exists a stable 2-coloring of pairs $f : [\omega]^2 \rightarrow 2$ and a family Y of infinite sets such that no $(f \oplus P)$ -computable set is p -homogeneous for f for any $P \in Y$, and every stable 2-coloring of pairs $f' : [\omega]^2 \rightarrow 2$ computable from f has either an $(f \oplus P)$ -computable increasing p -homogeneous set for some $P \in Y$, or if not then some increasing p -homogeneous set which does not compute a set p -homogeneous for f .*

Before proving Theorem 4.1, we observe that the Second Main Theorem is a direct consequence. The explanation is the same as that given in section 3 for the deduction of the First Main Theorem from Theorem 3.1.

We recall that \mathbb{C} is the following notion of forcing: a condition is an ordered triple $p = \langle \sigma^p, l^p, |p| \rangle$ where $|p| \in \omega$, $\sigma^p : [|p|]^2 \rightarrow 2$, $l^p : |p| \rightarrow 2 \times \omega$, and $l^p(x) = \langle i, z \rangle$ implies that if $\sigma^p(x, y)$ is defined and $y \geq z$ then $\sigma^p(x, y) = i$.

Lemma 4.2. If $p_0 \geq p_1 \geq p_2 \geq \dots$ is a sequence of \mathbb{C} -conditions which is 3-generic relative to some set $P \subseteq \omega$, and if $f = \bigcup_s \sigma^{p_s} : [\omega]^2 \rightarrow 2$, then $f \oplus P$ does not compute a p -homogeneous set for f .

Proof of Lemma 4.2. This follows from Lemma 3.3, since if $f \oplus P$ were to compute a p -homogeneous set for f , then from that set together with f one could compute a homogeneous set for f . \square

Proof of Theorem 4.1. We build

- a sequence of \mathbb{C} -conditions $p_0 \geq p_1 \geq p_2 \geq \dots$ with $\lim_s |p_s| = \infty$;
- sequences of finite sets (initial segments of increasing p -homogeneous sets) $H_{j,0}^\Phi \subseteq H_{j,1}^\Phi \subseteq H_{j,2}^\Phi \subseteq \dots$ for each Turing functional Φ and each $j < 2$;
- a sequence of infinite sets (reservoirs) $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ with $H_{j,s}^\Phi < I_s$ for each Φ, j, s ;
- a sequence of finite families $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ of infinite subsets of ω ;

and we define $f = \bigcup_s \sigma^{p_s}$, $H_j^\Phi = \bigcup_s H_{j,s}^\Phi$ for each $j < 2$, and $Y = \bigcup_s Y_s$. The construction will ensure the following requirements, for all $i \in \omega$ and Turing functionals Φ, Γ, Δ :

- \mathcal{P}_i : the sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ is 3-generic relative to each $P \in Y_i$;
- $\mathcal{Q}_{\Phi,i}$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$ -computable increasing p -homogeneous set for some $P \in Y$ or else both H_0^Φ and H_1^Φ are infinite in both columns;
- $\mathcal{R}_{\Gamma,\Delta}^\Phi$: if Φ^f is a stable 2-coloring of pairs, it either has an $(f \oplus P)$ -computable increasing p -homogeneous set for some $P \in Y$; or else if $\Gamma^{H_0^\Phi}$ defines an infinite set then this set is not p -homogeneous for f ; or else if $\Delta^{H_1^\Phi}$ defines an infinite set then this set is not p -homogeneous for f .

Construction.

Devote infinitely many stages $s \in \omega$ to each requirement. Let p_0 be any condition with $|p_s| = 0$. For each Φ let $H_{0,0}^\Phi = H_{1,0}^\Phi = \emptyset$, and let $I_0 = \omega$ and $Y_0 = \emptyset$. At stage $s + 1$ assume by way of induction that we have p_s , $H_{j,s}^\Phi$ for $j < 2$ and all Φ , I_s , and Y_s and assume that if $H_{j,s}^\Phi$ is nonempty for some j and Φ , then p_s forces that Φ^f is a stable coloring of pairs and that for $x < y$, $\Phi^f(x, y) = j$ whenever $2x, 2y + 1 \in H_{j,s}^\Phi$ or when $y \in I_s$ and $2x \in H_{j,s}^\Phi$. At the end of a stage any of p_{s+1} , $H_{j,s+1}^\Phi$, I_{s+1} , or Y_{s+1} not yet defined should be taken to be identical to p_s , $H_{j,s}^\Phi$, I_s , or Y_s , respectively.

\mathcal{P} requirements. Suppose s is dedicated to requirement \mathcal{P}_i for some $i < s$ and that it is the $\langle n, m \rangle^{\text{th}}$ such stage. If $n > |Y_i|$ do nothing. Otherwise, let P be the n^{th} member of the family Y_i in some fixed enumeration and let W be the m^{th} $\Sigma_3^0(P)$ set in some fixed enumeration. If p_s has an extension q in W , fix q and

let $p_{s+1} = q$, so that $p_0 \geq p_1 \geq \dots \geq p_s \geq p_{s+1} \geq \dots$ meets W . Otherwise, do nothing, and $p_0 \geq p_1 \geq \dots \geq p_s \geq p_{s+1} \geq \dots$ avoids W .

\mathcal{Q} requirements. Suppose s is dedicated to $\mathcal{Q}_{\Phi, i}$. Without loss of generality we assume that p_s decides whether or not Φ^f is a stable 2-coloring of pairs. If p_s forces that Φ^f is not such a coloring, do nothing. Otherwise, we consider two cases.

- If for some $j < 2$ and $k \in \omega$ there is no extension of p_s which forces that $\lim_u \Phi^f(x, u) = j$ for some $x \geq k$ in I_s , then $P = \{x \in I_s : x \geq k\}$ is limit homogeneous for Φ^f with color $1 - j$, and so $(f \oplus P)$ computes an increasing p -homogeneous set for Φ^f . We set $Y_{s+1} = Y_s \cup \{P\}$. This satisfies the requirement.
- If no such j, k exist, then there are numbers $x_{00}, x_{01}, x_{10}, x_{11} \in I_s$ with $x_{00} < x_{01}$ and $x_{10} < x_{11}$ and an extension of p_s forcing that $H_{j,s}^\Phi \cup \{2x_{j0}, 2x_{j1} + 1\}$ is finite increasing p -homogeneous for Φ^f and $\lim_u \Phi^f(x_{ji}, u) = j$ for each $i, j < 2$. In this case let p_{s+1} be such an extension of p_s , let $H_{j,s+1}^\Phi = H_{j,s}^\Phi \cup \{2x_{j0}, 2x_{j1} + 1\}$, and let $I_{s+1} = \{x : m < x \in I_s\}$ where m is greater than the stabilization points under Φ^f of every element of $H_{0,s+1}^\Phi \cup H_{1,s+1}^\Phi$. Observe that both columns have been extended by one element.

\mathcal{R} requirements. Suppose s is dedicated to $\mathcal{R}_{\Gamma, \Delta}^\Phi$ and assume that p_s forces that Φ^f is a stable coloring of pairs. As before, the goal of this requirement is to extend by some finite set either the initial segment of H_0^Φ or the initial segment of H_1^Φ that we have constructed so far, subject to the following condition: if we extend H_0^Φ , then Γ does not compute a p -homogeneous set for f from any further extension of H_0^Φ ; and if on the other hand we extend H_1^Φ , then Δ does not compute a p -homogeneous set for f from any further extension of H_1^Φ . Just as in the proof of Theorem 3.1, the finite set by which we extend either H_0^Φ or H_1^Φ will be obtained as a subset of the range of a path through an infinitely branching tree. When we have so extended one of H_0^Φ, H_1^Φ , we will say, as before, that we have successfully diagonalized against such computations.

The means by which we accomplish the diagonalization here will be slightly more involved. There will again be two numbers, say a and b , and two finite sets F_L and F_R , all arising from a tree labeling construction, about which we know either that

$$\Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(2a + 1) \downarrow = \Gamma^{H_0^\Phi \cup (F_L \oplus F_R)}(2b) \downarrow = 1$$

or else that

$$\Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(2a + 1) \downarrow = \Delta^{H_1^\Phi \cup (F_L \oplus F_R)}(2b) \downarrow = 1.$$

Let us suppose we know the first. In other words, we know that if we extend H_0^Φ by $F_L \oplus F_R$, then the set computed from $H_0^\Phi \cup (F_L \oplus F_R)$ by Γ will contain $2a + 1$ and $2b$. When we view the set so computed as having two columns (i.e. as being the join of two sets), this means that a appears in the right-hand column and b appears in the left-hand column. Thus to ensure that the set so computed is not p -homogeneous for f , we will choose as before an extension q of the condition p_s such that $\sigma^q(a, b)$, $(l^q(a))_0$, and $(l^q(b))_0$ are not all equal. This will guarantee that no set containing both a and b is p -homogeneous for the coloring eventually obtained by extending σ^q .

As in the proof of Theorem 3.1, the construction divides into Case B and Case A, whose description is unchanged. That is, in Case B we suppose there is an $i < 2$,

a condition $q \leq p_s$, an infinite set $I \subseteq I_s$, and a set $Q = \{\langle x, a_x, b_x \rangle : x \in I, a_x, b_x \in \omega\}$ such that there is exactly one triple in Q with first element x for each $x \in I$, $a_x < b_x$, and such that infinitely many numbers $b_x \in \omega$ appear as third elements of triples in Q ; and we suppose these have the following property: for any $x \in I$ and any extension r of q such that $\sigma^r(a_x, b_x)$, $(l^r(a_x))_0$, and $(l^r(b_x))_0$ are not all equal will force that $\lim_u \Phi^f(x, u) = i$. In Case A we suppose this is not so.

Case A. For this proof we need again to modify slightly the definition of tree labeling with two labels. We now say that a nonempty string $\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \text{ran}(\alpha^\#)$ and no $b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(2a + 1) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(2b) \downarrow = 1.$$

With this modification of the tree labeling definition in hand, we now as before let $T_0 = T(|p_s|, \Gamma, H_{0,s}^\Phi, I_s)$. If T_0 is not well-founded then let I_{s+1} be the range of an infinite path through T_0 . Observe that in this case the requirement is satisfied. If T_0 is well-founded, then let T_0^L be the labeled subtree of T_0 .

We now try to define two sequences, conditions

$$p_s \geq q_0 \geq q_1 \geq q_2 \geq \dots$$

and nodes of T_0^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \dots$$

where for all $j \geq 0$, α_{j+1} is a successor of α_j and for all $j \geq 0$ the condition q_j forces that

$$\lim_u \Phi^f(x, u) = 0$$

for all $x \in \text{ran}(\alpha_j)$. From here the proof is exactly the same as that of Case A in Theorem 3.1.

Case B. Fix an $i < 2$, a condition $q \leq p_s$, an infinite set $I \subseteq I_s$, and a set $Q = \{\langle x, a_x, b_x \rangle : x \in I, a_x, b_x \in \omega\}$ such that there is exactly one triple in Q with first element x for each $x \in I$; such that infinitely many numbers $b_x \in \omega$ appear as third elements of triples in Q ; and such that for any $x \in I$ and any extension r of q such that $\sigma^r(a_x, b_x)$, $(l^r(a_x))_0$, and $(l^r(b_x))_0$ are not all equal will force that $\lim_u \Phi^f(x, u) = i$. Suppose without loss of generality that $i = 1$. Thus for any $x \in I$, any forcing extension which presses the button of x forces that $\lim_u \Phi^f(x, u) = 1$.

Here again we must modify the tree labeling method, varying tree labeling with three labels. The definition of $T(k, \Gamma, H, I)$ is for this case changed to the following. $\emptyset \in T(k, \Gamma, H, I)$ and for a nonempty string α , $\alpha \in T(k, \Gamma, H, I)$ if $\alpha \in I^{<\omega}$ is increasing and there are no finite $F_L, F_R \subseteq \text{ran}(\alpha^\#)$ and no $c > b > a \geq k$ such that

$$\Gamma^{H \cup (F_L \oplus F_R)}(2a + 1) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(2b) \downarrow = \Gamma^{H \cup (F_L \oplus F_R)}(2c + 1) \downarrow = 1.$$

The method for labeling the nodes of $T(k, \Gamma, H, I)$ extends the method from tree labeling with two labels in the natural way, as does the method for selecting the nodes of the labeled subtree $T^L(k, \Gamma, H, I)$. Remark 2.3 applies *mutatis mutandis*. Finally, we use the revised the definition of a transition node from Case B in the proof of the previous theorem: a node $\alpha \in T^L$ is called a transition node if the symbol ∞ appears in the label of α , and appears strictly fewer times *and no more than once* in the label of each successor of α .

Now let $T_1 = T(|q|, \Delta, H_{1,s}^\Phi, I_s)$. If T_1 is not well founded then let I_{s+1} be the range of an infinite path through T_1 . Observe that in this case the requirement is satisfied. If T_1 is well founded, let T_1^L be the labeled subtree of T_1 .

We now try as in the proof of Theorem 3.1 to define two sequences, conditions

$$q \geq q_0 \geq q_1 \geq q_2 \geq \dots$$

and nodes of T_1^L

$$\emptyset = \alpha_0 \preceq \alpha_1 \preceq \alpha_2 \preceq \dots$$

where for all $j \geq 0$, α_{j+1} is successor of α_j and for all $j \geq 0$ the condition q_j forces that

$$\lim_u \Phi^f(x, u) = 0$$

for all $x \in \text{ran}(\alpha_j)$. From here the proof is exactly the same as that of Theorem 3.1 up until Case B.2.3, where the diagonalization strategy changes to reflect the different combinatorics of the present theorem. We therefore refer the reader to the previous proof, and restart this proof at Case B.2.3 below.

Recalling the function Sort defined in Case B.2.3 of the previous proof, we make a new definition. If $\alpha \in T_1^L$ is not terminal and $x, y \in \text{ran}(\alpha)$, we define the following properties:

- *Configuration I* holds if there is $F_L \oplus F_R \in \text{Sort}(\alpha)$ such that $x \in F_R$ and $y \in F_L$.
- *Configuration II* holds if there is $F_L \oplus F_R \in \text{Sort}(\alpha)$ such that $x, y \in F_L$ or $x, y \in F_R$ or one of x, y appears in neither of F_L, F_R .
- *Configuration III* holds if there is $F_L \oplus F_R \in \text{Sort}(\alpha)$ such that $x \in F_L$ and $y \in F_R$.

We now attend to Case B.2.3.

Case B.2.3. $\langle a, \infty, \infty \rangle \rightarrow \langle a, b, \infty \rangle$

If α_n has label $\langle a^*, \infty, \infty \rangle$ and every successor of α_n has a label in which the symbol ∞ appears exactly once, then we let

$$P = \{y : \alpha_n * y \in S \wedge \forall j < n (b > b_{\alpha_n(j)}) \wedge b > |q_n|\},$$

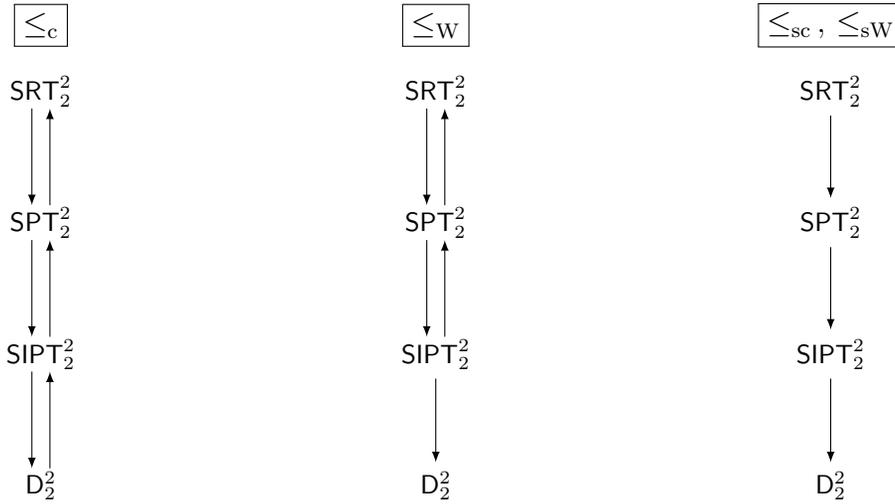
where b here denotes the second entry in the label of $\alpha_n * y$ and $b_{\alpha_n(j)}$ denotes the button of $\alpha_n(j)$. Whether there is much work to be done in this case depends on whether the first node to have label $\langle a^*, \infty, \infty \rangle$ was or was not the root node of T_1^L . Formally, suppose k is the least index such that α_k has a label in which the symbol ∞ appears exactly twice.

- If, on the one hand, $\alpha_k = \alpha_0 = \emptyset$, then we choose any $y^* \in P$ and any extension r of q_n which presses b_{y^*} and is such that, if the label of $\alpha_n * y^*$ is $\langle a^*, b^*, \infty \rangle$, then $\sigma^r(a^*, b^*)$, $(l^r(a^*))_0$, and $(l^r(b^*))_0$ are not all equal. In this case let $q_{n+1} = r$ and $\alpha_{n+1} = \gamma$ for any $\gamma \in S$ having $\gamma(n) = y^*$ with label $\langle a^*, b^*, \infty \rangle$.
- If, on the other hand, $k > 0$ and $\alpha_k \neq \emptyset$, then there is more work to do. Suppose that $\alpha_k(k-1) = x^*$. Let $P' \subseteq P$ contain precisely the elements y of P such that either Configuration I or Configuration II holds between x^* and y . Either $P' \neq \emptyset$ and we complete the diagonalization in this case, or else $P' = \emptyset$ and we wait until the next case to diagonalize, but we are then guaranteed to succeed.

We complete stage s as follows. If added some set P to Y , or if we defined I_{s+1} to be the range of an infinite path through T_0 or T_1 , we are done. Otherwise, we succeeded either in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_0^L or else in defining α_{n+1} for each non-terminal α_n in the sequence of nodes through T_1^L ; say we succeeded in defining the sequence of nodes in T_0^L . This tree was in this case well-founded, so for some i , α_n was terminal. Then from the definition of the tree, there are some $F_L, F_R \subseteq \text{ran}(\alpha_n)$ such that $\Gamma^{H_{0,s}^\Phi \cup (F_L \oplus F_R)}(2a+1) \downarrow = \Gamma^{H_{0,s}^\Phi \cup (F_L \oplus F_R)}(2b) \downarrow = \Gamma^{H_{0,s}^\Phi \cup (F_L \oplus F_R)}(2c+1) \downarrow = 1$ for some unequal $a, b, c \geq |p_s|$, say with use u . Let $p_{s+1} = q_i$, $H_{0,s+1}^\Phi = H_{0,s}^\Phi \cup F_L \oplus F_R$, and $I_{s+1} = \{x \in I_s : x > u\}$. \square

5. SUMMARY

The diagram below records all the reductions between the principles studied in this paper. In the diagram, we write $Q \rightarrow P$ to mean that problem P reduces to problem Q in the indicated sense.



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