EFFECTIVE COMPLETENESS FOR S4.3.1-THEORIES WITH RESPECT TO DISCRETE LINEAR MODELS

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Abstract. The computable model theory of modal logic was initiated by Suman Ganguli and Anil Nerode in [4]. They use an effective Henkin-type construction to effectivize various completeness theorems from classical modal logic. This construction has the feature of only producing models whose frames can be obtained by adding edges to a tree digraph. Consequently, this construction cannot prove an effective version of a well-known completeness theorem which states that every S4.3.1-theory has a model whose accessibility relation is a linear order of order type ω. We prove an effectivization of that theorem by means of a new construction adapted from that of Ganguli and Nerode.

1. Introduction

First-order modal logic has a rich model theory [2, 6] and is of considerable interest in philosophy [3], linguistics [7], and mathematics and computer science [6]. In view of these considerations, modal logic is an attractive candidate for a program of research in computable model theory. This program of research was initiated by Ganguli and Nerode in [4], who proved effective versions not only of the standard modal completeness theorem but of several of the variations of the completeness theorem unique to modal logic which guarantee for each decidable theory of a specified class a decidable model of a specified class. Namely, they proved the following theorems:

- every consistent, decidable theory has a decidable model;
- every consistent, decidable T-theory has a reflexive decidable model;
- every consistent, decidable K4-theory has a transitive decidable model;
- every consistent, decidable S4-theory has a reflexive and transitive decidable model;
- every consistent, decidable K5-theory has a transitive and Euclidean decidable model; and

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- every consistent, decidable S5-theory has a reflexive, transitive, and symmetric decidable model.

(See Definition 2.7 for the definition of these classes of theories and the classes of models they are intended to describe.) In addition, Ganguli and Nerode proved that their Henkin-type construction is robust in the following ways: the theorems remain true whether decidability of models is taken to require the decidability of truth or the decidability of global truth (simultaneous truth in every world), and they remain true whether the models are taken to have varying domains, constant domains, or monotonic domains. However, the construction in every case builds a branching tree of worlds to serve as the frame of the decidable model; the specialized theorems enumerated above are obtained by effectively adding edges to the accessibility relation (see Definition 2.1). This raises the question of whether these Henkin-type constructions are topologically robust, that is, whether a Henkin-type construction can be carried out to prove effective completeness theorems involving models whose accessibility relation cannot be obtained by adding edges to a branching tree.

We prove that in fact a Henkin-type construction, suitably modified, can be used to prove an effective completeness theorem involving models whose accessibility relations does not branch at all, yielding the following new theorem.

Theorem 1.1 (Effective Completeness for S4.3.1)**.** *Every decidable S4.3.1-theory has a decidable linear model of order type ω.*

This paper is organized as follows. In §2.1 we we give a concise but thorough overview of the modal logic necessary to our proof. In §2.2 we will introduce the system S4.3.1, including an exposition of its model theory which to our knowledge has never appeared previously. In particular, we correct a longstanding error due to Prior [9]. In §3 we prove Theorem 1.1. The proof depends on a Henkin-type construction which produces linear models of order type ω. The objects defined at each finite stage of the construction will be finite structures which approximate a modal model in the same way that finite conjunctions approximate the diagram of a classical first-order model in the classical Henkin construction. Because modal operators quantify over possible worlds in much the same way that first-order quantifiers quantify over elements in the domain, we will have to introduce two sorts of Henkin witnesses: elements of the domain to witness existential sentences, as in the classical construction; and possible worlds to witness possible sentences (i.e. sentences of the form $\Diamond\theta$). Since we wish to construct linear models, we will need to take care that the worlds we introduce as this sort of Henkin witness maintain the linearity of the structure; that is, as noted above, that the accessibility relation does not branch.

2. Preliminaries

2.1. **Modal Logic.**

We begin with a standard description of first-order modal logic. For a detailed introduction, see for example [2].

2.1.1. *Syntax.* A first-order modal language $\mathcal{L}$ is obtained by adding to a classical first-order language $\mathcal{L}^-$ the modal symbols $\Diamond$ and $\Box$.[1] The atomic formulae are

[1]For simplicity, we omit equality from the language. The only modification that needs to be made to our construction if equality is present is the modification one would expect in any

those of $\mathcal{L}^-$, and the well-formed formulae are obtained by adding to the standard recursive definition the rule:

- if ϕ is a wff, then so are $\Diamond\phi$ and $\Box\phi$.

Informally, $\Diamond\phi$ asserts that ϕ is possibly true while $\Box\phi$ asserts that ϕ is necessarily true. The sense of the words "possibly" and "necessarily" can vary greatly (see for example [2]) but is standardly formalized by a possible worlds semantics, given below. In terms of possible worlds, $\Diamond\phi$ means that there exists a possible world where ϕ is true, while $\Box\phi$ means that ϕ is true at every possible world.

2.1.2. *Semantics.* For the following definitions, fix a first-order modal language $\mathcal{L}$.

Definition 2.1. *A* Kripke frame *is a pair $F = (W, R)$, where W is a nonempty set of possible worlds and $R \subseteq W \times W$. R is called the* accessibility relation *of F.*

The accessibility relation is meant to formalize the idea that one world is possible *from* another: $(w, w') \in R$ means that w' is possible or "accessible" from w. In keeping with standard conventions, we identify a possible world with the set of sentences true at that world.

Definition 2.2. *A* (constant domain) Kripke model[2] *is a quadruple (W, R, D, I), where (W, R) is a Kripke frame, D is a nonempty set (the* domain *of the model), and I is an* interpretation *of the relations and constants of $\mathcal{L}$ in D having the following properties:*

- *for c a constant symbol, $I(c) \in D$;*
- *for $w \in W$ and P an n-ary predicate symbol, $I(w, P) \in D^n$.*

Given a Kripke model $M = (W_M, R_M, D_M, I_M)$, it is convenient to extend $\mathcal{L}$ to a new language $\mathcal{L}^M$ by adding for each $d \in D_M$ a constant c_d naming d. Then we expand M to an $\mathcal{L}^M$-model by taking $I_M(c_d) = d$ for each $d \in D_M$. This allows us more easily to state the following definition.

Definition 2.3 (Truth at a World)**.** *If w is a world in the Kripke model M, we write $(M, w) \vDash \phi$ when ϕ is true at the world w in M. Truth is defined inductively on the structure of ϕ:*

- *$(M, w) \vDash P(c_0, \dots, c_n)$ iff $(I(c_0), \dots, I(c_n)) \in I(w, P)$;*
- *$(M, w) \vDash \neg\phi$ iff $(M, w) \nvDash \phi$;*
- *$(M, w) \vDash \phi \wedge \psi$ iff $(M, w) \vDash \phi$ and $(M, w) \vDash \psi$;*
- *$(M, w) \vDash \exists x \phi(x)$ iff $(M, w) \vDash \phi(c_d)$ for some $d \in D_M$;*
- *$(M, w) \vDash \Diamond\phi$ iff $(M, w') \vDash \phi$ for some $w' \in W_M$ such that $(w, w') \in R_M$;*
- *$(M, w) \vDash \Box\phi$ iff $(M, w') \vDash \phi$ for all $w' \in W_M$ such that $(w, w') \in R_M$.*

Definition 2.4. *A sentence ϕ is a* theorem *of M, written $M \vDash \phi$, if $(M, w) \vDash \phi$ for every $w \in W_M$.*

If T is a set of sentences, we write $M \vDash T$ to mean that $M \vDash \phi$ for each $\phi \in T$.

Definition 2.5. *We say that T* entails *ϕ, and write $T \vDash \phi$, if for every Kripke model $M \vDash T$, $M \vDash \phi$.*

Henkin construction: the domain of the constructed model should be taken to be the set of Henkin witnesses modulo equality. For the details of this modification in the modal logic case, see [4].

[2]We consider only constant domain models, but the construction we give may be adapted to varying domain models and to monotonic domain models using, *mutatis mutandis*, the modifications for those cases described in [4].

2.1.3. *Deduction.* The standard system of deduction in first-order modal logic is *normal modal logic*, denoted K, which consists of all the inference rules and axiom schema of first-order logic,[3] plus the following rule and scheme:

- **Rule of Necessitation:** if ϕ is a theorem then $\Box\phi$ is a theorem.
- **Axiom of Distribution:** $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

The soundness and completeness theorems hold between the semantics of Kripke models given above and this deductive system.

2.1.4. *Theories.*

Definition 2.6. *A* theory *is a deductively closed set of sentences.*

Modal theories are classified according to whether they contain certain axiom schema (or, equivalently, are closed under a new notion of deduction obtained by including those axiom schema as logical axioms).

Definition 2.7. *We define some standard classes of theories.*

- K-theories *are just theories, with no additional axioms required.*
- T-theories *contain the scheme* $\Box\phi \to \phi$*.*
- K4-theories *contain the scheme* $\Box\phi \to \Box\Box\phi$*.*
- K5-theories *contain the scheme* $\Diamond\phi \to \Box\Diamond\phi$*.*
- S4-theories *contain the schemes of both T and K4.*
- S5-theories *contain the schemes of S4 and K5.*

These classes of theories are usually referred to as *systems*. As stated in the introduction, effective completeness theorems for each of these systems were proved in [4].

2.2. **The System S4.3.1.** Arthur Prior [8] introduced the modal system D to describe discrete time, i.e. time seen as a linear sequence of moments such that for each moment, there is an earliest (if any) future moment and a latest (if any) past moment. Such a conception of time, and hence the corresponding modal logic, has proven useful in computer science (see e.g. [5]). System D was later renamed to system S4.3.1 by Sobociński [11]. As the name D presently designates another modal system, the system of discrete time has retained Sobociński's nomenclature. System S4.3.1 is obtained from K by adding the following axiom schema:

- T: $\Box\phi \to \phi$
- 4: $\Box\phi \to \Box\Box\phi$
- D2: $\Box(\Box\phi \to \psi) \vee \Box(\Box\psi \to \phi)$
- N1: $\Box(\Box(\phi \to \Box\phi) \to \phi) \to (\Diamond\Box\phi \to \phi)$

Given axiom schema T and 4, axiom scheme D2 describes *linear* models, i.e. models whose accessibility relation is a linear order; given the previous three schema, N1 describes *discrete* models, i.e. models whose accessibility relation has the order type of an infinite subset of the integers.

In order to clarify them, we will verify the claims of the preceding paragraph informally. Consider the temporal interpretation of $\Box\phi$ as asserting that "ϕ is now and always will be true," i.e. that ϕ is true at the present moment and is true at all presently possible future moments. On that reading, we have the following understanding of the axiom schema D2 and N1.

[3] In Kripke models which are not constant domain models, one classical inference rule must be omitted: $\phi(x)$ and $\forall x\phi(x)$ are not in general equivalent.

- D2
  $\Box(\Box\phi \to \psi)$ means that whenever ϕ is forever true, ψ is true; $\Box(\Box\psi \to \phi)$ means that whenever ψ is forever true, ϕ is true. For this to be false at some world (moment) in a transitive, reflexive model, there would need to be a possible future moment w_1 at which ϕ was forever true while ψ was false, as well as a possible future moment w_2 at which ψ was forever true while ϕ is false. This is impossible if $(w_1, w_2) \in R$ or $(w_2, w_1) \in R$, so the accessibility relation cannot be total. So given T and 4, to falsify D2 we require that the accessibility relation not be a linear order.
- N1
  $\Diamond\Box\phi \to \phi$ means that if sooner or later ϕ is forever true, then ϕ is presently true; $\Box(\Box(\phi \to \Box\phi) \to \phi)$ means that it is forever true that if whenever ϕ becomes true it stays true forever, then ϕ is true. For this to be false at some world in a model whose accessibility relation is a linear order, the following would have to obtain: ϕ is false, sooner or later ϕ is forever true, and at every future moment, either ϕ is true or ϕ will become true without being forever true. Given that the accessibility relation is a reflexive linear order, this requires that there be in our model a moment (world) at which ϕ is forever true, but prior to which there is no latest moment at which ϕ is true and no latest moment at which ϕ is false. So given T, 4, and D2, to falsify N1 we require a subset of the accessibility relation to be dense.[4]

In light of the above considerations, the following theorem is not surprising. The proof of the theorem was first given by Bull [1] using an algebraic semantics for modal logic rather than the now standard Kripke possible world semantics outlined in this paper; Segerberg [10] later supplied a proof using Kripke semantics.

Theorem 2.8 (Completeness for S4.3.1)**.** *The modal system S4.3.1 is complete with respect to the class of models whose accessibility relation is a linear order of order type ω.*

Completeness theorems like this one in modal logic are usually proven using a noneffective canonical model construction. While the usual proof strategy fails in the case of Theorem 2.8 [2], its proof still relies on the canonical model construction [10]. Moreover, any attempt to effectivize Theorem 2.8 using the effective Henkin construction of [4] fails because that construction produces nonlinear models. In fact, whenever a sentence of the form $\Diamond\phi$ is added to a world w during the construction, a new world w' such that $(w, w') \in R$ is added to the model; and if w'' is another such world, then $(w', w'') \notin R$.

Therefore in order to effectivize Theorem 2.8, we require a new Henkin construction. We supply such a construction in the next section.

[4]Historical note: the informal proof of this fact usually cited is that of Prior in [9], which is incorrect. Prior proposes a countermodel in which ϕ is true at moment M and at every moment thereafter, ϕ is false at every moment preceding M, and there is no latest moment preceding M. Such a model in fact validates the instance of N1: at worlds where ϕ is false, both the antecedent and the consequent of the axiom are false; and at worlds where ϕ is true, both the antecedent and the consequent are true.

3. Construction

Recall our goal: to prove that every decidable S4.3.1-theory has a decidable model whose accessibility relation is of order type ω. Because we will be referring repeatedly to such models, it is convenient to give them a less cumbersome name.

Definition 3.1. *A Kripke model whose accessibility relation is a linear order of order type ω is called a* discrete linear model.

Our construction will build such a model in stages. At each finite stage, we will have built a sort of finite approximation to the model. The following definition makes this precise.

Definition 3.2.

(1) A finite Kripke diagram (FKD) *for a modal language $\mathcal{L}$ is an ordered pair $(W = \{w_0, \dots, w_n\}, R)$, where W is a finite set of finite worlds (sets of $\mathcal{L}$-sentences) and R a binary relation on W. A* linear FKD *is a FKD where*
$$\{(w_i, w_j) : i + 1 = j\} \subseteq R \subseteq \{(w_i, w_j) : i \leq j\}.$$
(2) Given a linear FKD $D = (W = \{w_0, \dots, w_n\}, R)$, define
$$\Psi_n = \bigwedge\{\phi \mid \phi \in w_n\}$$
and define by induction
$$\Psi_i = \bigwedge(\{\phi \mid \phi \in w_i\} \cup \{\Diamond\Psi_j \mid (w_i, w_j) \in R\})$$
for $0 \leq i < n$. We say that $\Psi^D := \Psi_0$ is the representing formula of D.

At successive stages of the construction we will extend linear FKDs to larger linear FKDs, with the intention that the desired model can be defined as a limit of this construction. We therefore need to know whether these extensions are "consistent" in the sense that we may in fact define such a limit model. It turns out that the notion of consistency we need our FKDs to have is something a little short of requiring them to embed into a model. The correct notion is supplied in the following definition.

Definition 3.3.

(1) If $D = (W_D, R_D)$ is a linear FKD, a discrete linear model $M = (W_M, R_M, D_M, I_M)$ in the same language is said to witness D by f *if $f : W_D \to W_M$ is a function such that $(w_i, w_j) \in R_D$ implies $(f(w_i), f(w_j)) \in R_M$ and $\phi \in w_i$ implies $(M, f(w_i)) \vDash \phi$.*
(2) If T is an S4.3.1-theory, a linear FKD D is T-consistent *if there exists a discrete linear model M of T which witnesses D.*

Given a decidable S4.3.1-theory T, we can determine effectively whether a given linear FKD is T-consistent. This is the import of the next lemma. Before we state the lemma, however, we have need of a new notation.

Definition 3.4. *If T is a theory and ϕ a sentence in the language of T, we write $T \vDash_l \phi$ if ϕ holds in every discrete linear model of T.*

Lemma 3.5 (Testing Lemma)**.** *For a linear FKD $D = (W_D = \{w_0, \dots, w_n\}, R_D)$ and a theory T, the following are equivalent:*

(1) D is T-consistent;

(2) $T \not\models_l \neg\Psi^D$.

Observe that since T is complete, if we take T to be decidable we can determine effectively whether (2) holds.

Proof.

- (1) $\Rightarrow$ (2).
  Suppose that D is T-consistent, and let M be a discrete linear model witnessing D by f. The proof proceeds by induction.

  By hypothesis, $(M, f(w_n)) \models \phi$ for each $\phi \in w_n$, whence $(M, f(w_n)) \models \Psi_n$. Now suppose by way of induction that $0 \leq i < n$ and $(M, f(w_{i+1})) \models \Psi_{i+1}$. Then since $(f(w_i), f(w_{i+1})) \in R_M$ and $(M, f(w_{i+1})) \models \Psi_{i+1}$, $(M, f(w_i)) \models \Diamond\Psi_{i+1}$. Since also $(M, f(w_i)) \models \Psi_i$ by the same reasoning applied above to w_n, it follows that $(M, f(w_i)) \models \Psi_i$.

  By induction, $(M, f(w_i)) \models \Psi_i$ for each $0 \leq i \leq n$, and in particular $(M, f(w_0)) \models \Psi^D$. Thus we have a model of T satisfying Ψ^D; i.e. $T \not\models_l \neg\Psi^D$.

- (2) $\Rightarrow$ (1).
  Suppose that $T \not\models_l \neg\Psi^D$, and let $M = (W_M, R_M, D_M, I_M)$ be a discrete model and $w \in W_M$ a world of M such that $(M, w) \models \Psi^D$. We will define a function $f : W_D \to W_M$ by induction such that M witnesses D by f.

  Set $f(w_0) = w$. By hypothesis, $(M, w) \models \Psi_0$. Now suppose by way of induction that we have defined $f(w_i)$ such that $(M, f(w_i)) \models \Psi_i$. Since $\Diamond\Psi_{i+1}$ is among the conjuncts of Ψ_i, $(M, f(w_i)) \models \Diamond\Psi_{i+1}$ and so there exists $u \in W_M$ such that $(f(w_i), u) \in R_M$ and $(M, u) \models \Psi_i$. Set $f(w_i) = u$ for some such u.

  Since by induction $(M, f(w_i)) \models \Psi_i$ for each $0 \leq i \leq n$, we observe that M witnesses D by f.

$\square$

It will greatly simplify the description of our construction if we define a notation whereby if D is a FKD, $D + \{*\}$ denotes the modification of D by adding $*$ to it, in the following precise sense.

Definition 3.6. *If $D = (W_D = \{w_0^D, \ldots, w_n^D\}, R_D)$ is a FKD and ϕ a sentence in the language of D, define the following notations.*

- *For $0 \leq i \leq n$, let $D + \{\phi \in w_i\}$ denote the FKD $(W = \{w_0, \ldots, w_{n+1}\}, R_D)$, where $w_i = w_i^D \cup \{\phi\}$ and $w_j = w_j^D$ for $j \neq i$.*
- *For $0 \leq i \leq n+1$, let $D + \{w_i\}$ denote the FKD $(W = \{w_0, \ldots, w_{n+1}\}, R)$, where*

$$w_j = \begin{cases} w_j^D, & \textit{if } j < i; \\ w_i = \emptyset, & \textit{if } j = i; \\ w_{j-1}^D, & \textit{if } n+1 \geq j > i; \end{cases}$$

  and where

$$R = \{(w_j, w_k) : j + 1 = k \textit{ or } w_j = w_{j^*}^D, w_k = w_{k^*}^D, (w_{j^*}^D, w_{k^*}^D) \in R_D\}.$$

  Thus $D + \{w_i\}$ is obtained by splicing a new world into the i^{th} position among the worlds in D.
- *The two previous notations may be combined: for $0 \leq i \leq n+1$, let $D + \{w_i, \phi \in w_i\}$ denote $(D + \{w_i\}) + \{\phi \in w_i\}$.*

These notations may be repeated, so that $D + \{\phi \in w_i, \psi \in w_j\}$ denotes the FKD obtained from D by adding ϕ to w_i^D and ψ to w_j^D. With this notation in hand, we now begin the construction.

Construction 3.7. Let T be a consistent, decidable S4.3.1-theory in a first-order modal language $\mathcal{L}$. Fix a computable set of constants C absent from $\mathcal{L}$ and a computable enumeration $\phi_0, \phi_1, \phi_2, \ldots$ of sentences in the expanded language $\mathcal{L} \cup C$. Let $\pi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a computable enumeration of pairs with the property that for any given pair $(i, e) \in \mathbb{N} \times \mathbb{N}$, there are infinitely many $n \in \mathbb{N}$ such that $\pi(n) = (i, e)$.

We will construct a discrete linear $\mathcal{L}$-model M of T in stages. At stage -1 we define the linear FKD $D_0 = (\{w_0^0 = \emptyset\}, R_0 = \emptyset)$. At stage n we have by induction a linear FKD $D_n = (W_n = \{w_0^n, \ldots, w_p^n\}, R_n)$. We will construct a linear FKD D_{n+1} which extends D_n as follows. Compute $\pi(n) = (i, e)$. If $i > p$, merely update indices from n to $n+1$, i.e. set

$$D_{n+1} = (W_{n+1} = \{w_0^{n+1}, \ldots, w_p^{n+1}\}, R_{n+1}) = D_n,$$

and proceed to the next stage. If on the other hand $i \leq p$, then let D denote $D_n + \{\phi_e \in w_i^n\}$ and check (decidably) whether $T \vDash_l \neg\Psi^D$.

- If $T \vDash_l \neg\Psi^D$, set $D_{n+1} = D_n + \{\neg\phi_e \in w_i^n\}$ and update indices from n to $n+1$.
- If $T \nvDash_l \neg\Psi^D$, we consider several cases.
  - If $\phi_e = \exists x\, \theta(x)$, let $D_{n+1} = D_n + \{\phi_e, \theta(c_j) \in w_i^n\}$ where c_j is the least element of C not occurring in D_n, and update indices.
  - If $\phi_e = \Diamond\theta$, we examine the following FKDs in order:

$$D_n + \{\phi_e \in w_i^n, \theta \in w_i^n\},$$

$$D_n + \{\phi_e \in w_i^n, w_{i+1}, \theta \in w_{i+1}\},$$

$$D_n + \{\phi_e \in w_i^n, \theta \in w_{i+1}^n\},$$

$$D_n + \{\phi_e \in w_i^n, w_{i+2}, \theta \in w_{i+2}\},$$

$$D_n + \{\phi_e \in w_i^n, \theta \in w_{i+2}^n\},$$

$$\vdots$$

$$D_n + \{\phi_e \in w_i^n, w_{p-1}, \theta \in w_{p-1}\},$$

$$D_n + \{\phi_e \in w_i^n, \theta \in w_p^n\},$$

$$D_n + \{\phi_e \in w_i^n, w_{p+1}, \theta \in w_{p+1}\}.$$

    In each case we test whether $T \nvDash_l \neg\Psi^{\widehat{D}}$, where $\widehat{D}$ is the FKD in question. If this is the case, we set $D_{n+1} = \widehat{D}$ and update indices from n to $n+1$. If not, we proceed to the next item in the list. We are guaranteed to obtain a positive result from one of these tests because D_n is by induction T-consistent and T has a discrete linear model.
  - Otherwise, we set $D_{n+1} = D_n + \{\phi_e \in w_i^n\}$ and update indices.

Now we define a Kripke model $M = (W, R, D, I)$:

- $W = \left\{ \bigcup_{n=1}^{\infty} w_i^n : w_i^n \text{ occurs in some } D_n \right\}$;
- $R = \left\{ (w_i, w_j) : w_i^n, w_j^n \in W_n \text{ for some } n,\ i \leq j \right\}$;

- $D = C$;
- and for an n-ary relation symbol P and n-tuple $\bar{c}$ of elements of C, and any world $w_i \in W$,
$$\bar{c} \in I(w_i)(P) \Leftrightarrow P(\bar{c}) \in w_i.$$

Lemma 3.8 (Consistency Lemma). *At each stage n of the Construction 3.7, D_n is T-consistent.*

Proof. The proof is by induction. To say that D_0 is T-consistent is just to say that T is consistent, which holds by hypothesis. Now suppose that we have shown D_n to be T-consistent. We consider the possible cases.

- $T \vDash_l \neg\Psi^D$. In this case, we chose $D_{n+1} = D_n + \{\neg\phi_e \in w_i^n\}$. Let M be a discrete linear model witnessing D_n by f. By the Testing Lemma, D is not T-consistent, so M cannot witness D. Hence $(M, f(w_i^n)) \nvDash \phi_e$ and then $(M, f(w_i^n)) \vDash \neg\phi_e$. So M witnesses D_{n+1} by f (subject to updated indices), meaning D_{n+1} is T-consistent.
- $T \nvDash_l \neg\Psi^D$. Let $M = (W_M, R_M, D_M, I_M)$ be a discrete linear model witnessing D by f, so that in particular $(M, f(w_i^n)) \vDash \phi_e$. We consider the possible cases.
  - $\phi_e = \exists x\, \theta(x)$. Then since $(M, f(w_i^n)) \vDash \phi_e$, there is an element $a \in D_M$ such that $(M, f(w_i^n)) \vDash \theta(c_a)$. Then (W_M, R_M, D_M, I) witnesses D_{n+1} by f (subect to updated indices), where $I(c_j) = a$ and $I = I_M$ otherwise. So D_{n+1} is T-consistent.
  - $\phi_e = \Diamond\theta$. Since $(M, f(w_i^n)) \vDash \phi_e$, there is a world $u \in W_M$ such that $(f(w_i^n), u) \in R_M$ and $(M, u) \vDash \theta$. Fix such a world $u \in W_M$ and define $\hat{f}$ as follows.
$$\hat{f}\left(w_j^{n+1}\right) = \begin{cases} f\left(w_j^n\right), & \text{if } j < k; \\ u, & \text{if } j = k; \\ f\left(w_{j-1}^n\right), & \text{if } p+1 \geq j > k. \end{cases}$$
Then M witnesses D_{n+1} by $\hat{f}$.
  - Otherwise, $D_{n+1} = D$ with and so M witnesses D_{n+1} by f (subject to updated indices). So D_{n+1} is T-consistent.

$\square$

Lemma 3.9 (Closure Lemma). *Let T, C, M, etc. be as in Construction 3.7. Then for each $w_i \in W$:*

(1) For each e, exactly one of ϕ_e or $\neg\phi_e$ is in w_i.
(2) $(\phi \wedge \psi) \in w_i$ iff $\phi \in w_i$ and $\psi \in w_i$.
(3) $\exists x\, \theta(x) \in w_i$ iff tehere is some $c \in C$ such that $\theta(c) \in w_i$.
(4) $\Diamond\theta \in w_i$ iff there is some $w_j \in W$ such that $(w_i, w_j) \in R$ and $\theta \in w_j$.
(5) $\Box\theta \in w_i$ iff $\theta \in w_j$ for every $w_j \in W$ such that $(w_i, w_j) \in R$.
(6) $T \subseteq w_i$.

Proof.

(1) Let $n = (i, e)$ be such that $w_i^n \in W_n$. By Construction 3.7, either $\phi_e \in w_i^{n+1}$ or $\neg\phi_e \in w_i^{n+1}$ and so one or the other is in w_i. Now suppose by way of contradiction that $\phi_e, \neg\phi_e \in w_i$. Then $\phi_e, \neg\phi_e \in w_i^n$ for some n. But then D_n is not T-consistent, contrary to the Consistency Lemma.

(2)
- ($\Rightarrow$). Suppose by way of contradiction that $(\phi \wedge \psi) \in w_i$ but that one of ϕ, ψ is not in w_i. For definiteness suppose $\phi \notin w_i$. Then $\neg\phi \in w_i$, so $(\phi \wedge \psi), \neg\phi \in w_i^n$ for some n. But then D_n is not T-consistent. Hence $\phi \in w_i$.
- ($\Leftarrow$). Suppose by way of contradiction that $\phi, \psi \in w_i$ but that $(\phi \wedge \psi) \notin w_i$. Then $\neg(\phi \wedge \psi) \in w_i$, and so $\phi, \psi, \neg(\phi \wedge \psi) \in w_i^n$ for some n. But then D_n is not T-consistent.

(3)
- ($\Rightarrow$). Suppose $\exists x\, \theta(x) \in w_i$. Then at whichever stage n of Construction 3.7 $\exists x\, \theta(x)$ was added to w_i^n, $\theta(c)$ was added also, for some $c \in C$.
- ($\Leftarrow$). Suppose by way of contradiction that $\theta(c) \in w_i$ for some $c \in C$ but that $\exists x\, \theta(x) \notin w_i$. Then $\neg\exists x\, \theta(x) \in w_i$, so $\theta(c), \neg\exists x\, \theta(x) \in w_i^n$ for some n. But then D_n is not T-consistent.

(4)
- ($\Rightarrow$). Suppose $\Diamond\theta \in w_i$. Then at some stage n of Construction 3.7 $\Diamond\theta$ was added to w_i^n, and at the same stage a new world w_j^{n+1} was added such that $\theta \in w_j^{n+1}$ and $j \geq i$. Therefore $\theta \in w_j$ and $(w_i, w_j) \in R$.
- ($\Leftarrow$). Suppose by way of contradiction that $(w_i, w_j) \in R$ and $\theta \in w_j$, but that $\Diamond\theta \notin w_i$. Then $\neg\Diamond\theta \in w_i$, so $\theta \in w_j^n$, $\neg\Diamond\theta \in w_i^n$, and $i \leq j$ for some n. But then D_n is not T-consistent.

(5)
- Suppose by way of contradiction that $\Box\theta \in w_i$ but that $\theta \notin w_j$ for some $j \geq i$. Then $\Box\theta \in w_i^n, \neg\theta \in w_j^n$, and $i \leq j$ for some n. But then D_n is not T-consistent.
- Suppose $\Box\theta \notin w_i$. Then $\neg\Box\theta \in w_i$ and hence $\Diamond\neg\theta \in w_i$ (otherwise, $\neg\Box\theta, \neg\Diamond\neg\theta \in w_i^n$ for some n, implying D_n is not T-consistent). Therefore there is $j \geq i$ such that $\neg\theta \in w_j$, and therefore $\theta \notin w_j$.

(6) Suppose by way of contradiction that $\phi \in T$ but $\phi \notin w_i$. Then $\neg\phi \in w_i$, so $\neg\phi \in w_i^n$ for some n. But since any model M witnessing D_n by f is a model of T, this implies that D_n is not T-consistent.

□

Lemma 3.10 (Truth Lemma)**.** *Let T, C, M be as in Construction 3.7. Then for each $w_i \in W$ and each $\phi \in L \cup C$,*

$$(M, w_i) \vDash \phi \Leftrightarrow \phi \in w_i.$$

Proof. The proof is by induction. If ϕ is atomic, the equivalence in the lemma holds by definition of I. Now suppose that the equivalence holds for α, β. We consider the inductive cases.

- $(M, w_i) \vDash (\alpha \wedge \beta)$ iff $(M, w_i) \vDash \alpha$ and $(M, w_i) \vDash \beta$ iff $\alpha, \beta \in w_i$ by the inductive hypothesis, iff $(\alpha \wedge \beta) \in w_i$ by the Closure Lemma.
- $(M, w_i) \vDash \neg\alpha$ iff $(M, w_i) \nvDash \alpha$ iff $\alpha \notin w_i$ by the inductive hypothesis, iff $\neg\alpha \in w_i$ by the Closure Lemma.
- $(M, w_i) \vDash \exists x\, \alpha(x)$ iff $(M, w_i) \vDash \alpha(c)$ for some $c \in C$ iff $\alpha(c) \in w_i$ for some $c \in C$ by the inductive hypothesis, iff $\exists x\, \alpha(x) \in w_i$ by the Closure Lemma.
- $(M, w_i) \vDash \Diamond\alpha$ iff $(M, w_j) \vDash \alpha$ for some $j \geq i$ iff $\alpha \in w_j$ by the inductive hypothesis, iff $\Diamond\alpha \in w_i$ by the Closure Lemma.
- $(M, w_i) \vDash \Box\alpha$ iff $(M, w_j) \vDash \alpha$ for every $j \geq i$ iff $\alpha \in w_j$ for every $j \geq i$ by the inductive hypothesis, iff $\Box\alpha \in w_i$ by the Closure Lemma.

□

Corollary 3.11. *Let T, M be as in Construction 3.7. Then $M \vDash T$.*

Corollary 3.12. *For each $w_i \in W$ and each $\phi \in T$, $\phi \in w_i$ by the Closure Lemma and therefore $(M, w_i) \vDash \phi$ by the Truth Lemma.*

We are now in a position to prove Theorem 1.1.

Theorem 1.1 (Effective Completeness for S4.3.1)**.** *Every decidable S4.3.1-theory has a decidable linear model of order type ω.*

Proof of Theorem 1.1. Fix a consistent, decidable S4.3-theory T. Let M be the Kripke model produced by Construction 3.7.

- By Corollary 12, M is a model of T.
- Since $R = \{(w_i, w_j) : w_i^n, w_j^n \in W_n \text{ for some } n,\ i \leq j\}$, M is linear.
- Fix $w_i, w_j \in W$. Since $(w_i, w_j) \in R$ iff $i \leq j$, R is decidable.
- Fix $\phi \in L \cup C$ and $w_i \in W$. To decide whether $(M, w_i) \vDash \phi$, search through the enumeration $\phi_0, \phi_1, \ldots$ to find the index e such that $\phi = \phi_e$. Then search through the stages n of Construction 3.7 such that $\pi(n) = (i, e)$ to find the stage at which one of $\phi_e, \neg\phi_e$ is added to w_i^n. Then $\phi \in w_i$ iff ϕ_e is added to w_i^n, iff $(M, w_i) \vDash \phi$ by the Truth Lemma. Such an n can be found decidably, so truth at a world in M is decidable.

□

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